Practical guide to loop integration
Exercise Sheet 6

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Summer Term 2024
Dr Y. Ulrich

https://yannickulrich.com/loop-integration

Issued: 21 May 2024

Exercise 1: Multidimensional Frobenius method   In this problem we will explore the Frobenius method. Consider the following system of differential equations

$$\frac{\partial \vec f}{\partial x} = \underbrace{\begin{pmatrix} \frac{2-x}{1-x} & -\frac{1}{1-x} \\ -x & \frac{x}{1-x} \end{pmatrix}}_{M} \vec f\,. \tag{1}$$

A solution of this is given by

$$\vec f = \Big(\frac{\log(1-x)}{1-x}, \frac{1}{1-x} + \log(1-x)\Big)^T\,. \tag{2}$$

You may use this solution step-by-step to verify your working though in a real-life example, you would obviously not know it.

a) Construct the matrices $M^{(j)}$ for $j=0$, $j=1$, and $j=2$. Write down the matrices $\bar M$ and $\tilde M$

b) Find a vector $\vec c$ from the left-nullspace of $\tilde M$ such that $\vec c \tilde M = 0$.

c) Make an ansatz for $f_1 = x^r \sum x^i a_i$ and calculate $\vec c \cdot (f_1, \partial_x f_1, \partial_x^2 f_1)$. Find the biggest r, a_1, a_2, and a_3.

d) Construct $\vec f$ by inverting $\tilde M$.

SOLUTION: We begin by verifying that (2) is indeed a valid solution of (1).

a) Using the recursive definition

$$M^{(0)} = 1 \qquad \text{and} \qquad M^{(j)} = \frac{\partial M^{(j-1)}}{\partial x} - M^{(j-1)} M\,, \tag{3}$$

we have

$$M^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

$$M^{(1)} = M \tag{5}$$

$$M^{(2)} = \begin{pmatrix} \frac{5-3x}{(1-x)^2} & -\frac{3}{(1-x)^2} \\ \frac{1+x}{1-x} & \frac{1+x}{(1-x)^2} \end{pmatrix}\,. \tag{6}$$

With this we can just read of

$$\bar{M} = \begin{pmatrix} 1 & 0 \\ \frac{2-x}{1-x} & \frac{1}{x-1} \end{pmatrix} \tag{7}$$

$$\tilde{M} = \begin{pmatrix} 1 & 0 \\ \frac{2-x}{1-x} & \frac{1}{x-1} \\ \frac{5-3x}{(1-x)^2} & -\frac{3}{(1-x)^2} \end{pmatrix} . \tag{8}$$

At this point we may verify that

$$\begin{aligned} \frac{\partial^j \vec{f}}{\partial x^j} &= M^{(j)} \vec{f}, \\ \left(f_1, \partial_x f_1, \partial_x^2 f_1\right) &= \tilde{M} \vec{f}. \end{aligned} \tag{9}$$

b) It is easy to see that

$$\vec{c} = \begin{pmatrix} \frac{1}{1-2x+x^2} \\ \frac{-3+3x}{1-2x+x^2} \\ 1 \end{pmatrix} \tag{10}$$

is a valid and properly normalised solution. You may want to verify that $\vec{c} \cdot (f_1, \partial_x f_1, \partial_x^2 f_1) = 0$.

c) Expanding and collecting terms we have with $a_0 = 1$

$$x^r \left(\frac{(r-1)r}{x^2} + \mathcal{O}(x^{-1}) \right) = 0 . \tag{11}$$

The valid solutions are $r = 0$ or $r = 1$. We now expand higher with $r = 1$

$$(-3 + 2a_1) + (-2 - 6a_1 + 6a_2)x + (-1 - 5a_1 - 9a_2 + 12a_3)x^2 + \mathcal{O}(x^3) = 0 \tag{12}$$

to see that $a_1 = 3/2$, $a_2 = 11/6$, and $a_3 = 25/12$. We can also explitly verify that expanding (2) gives these expansions coefficients up to an overall sign.

d) The inverse matrix is

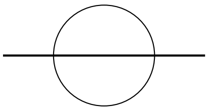
$$\tilde{M}^{-1} = \begin{pmatrix} 1 & 0 \\ 2 - x & -1 + x \end{pmatrix} . \tag{13}$$

And therefore $\vec{f} = \tilde{M}^{-1}(f_1, \partial_x f_1)^T$

$$f_2 = -1 - \frac{x^2}{2} - \frac{2x^3}{3} . \tag{14}$$

Exercise 2: Two-mass sunset diagram in $d = 2 - 2\epsilon$

Consider the following integral

$$I_{\alpha\beta\gamma} = \quad$$  $$= \int \frac{[\mathrm{d}k_1][\mathrm{d}k_2]}{\left[k_1^2 - m_1^2\right]^{\alpha}\left[k_2^2 - m_1^2\right]^{\beta}\left[(k_1+k_2+p)^2 - m_2^2\right]^{\gamma}}\,, \tag{15}$$

with $p^2 = m_2^2 \neq m_1^2$ in $d = 2 - 2\epsilon$ dimensions. We use the integrals

$$\vec{I} = \left(m_2^2\right)^{4-d}\epsilon^2\left(z^{-4}I_{220}, z^{-2}I_{202}, z^{-4}I_{211}, z^{-1}I_{112}\right)^T \tag{16}$$

and the variable $z = m_2/m_1$.

a) Show that the differential equation matrix is precanonical

$$\partial_z\vec{I} = \begin{pmatrix} \frac{4\epsilon}{z} & 0 & 0 & 0 \\ 0 & \frac{2\epsilon}{z} & 0 & 0 \\ \frac{1}{z(1-z^2)} & -\frac{1}{z(1-z^2)} & \frac{z^2+(6-4z^2)\epsilon}{z(1-z^2)} & -\frac{1+2\epsilon}{z^2(1-z^2)} \\ -\frac{z^2}{1-z^2} & \frac{1}{1-z^2} & -\frac{z^2(1+2\epsilon)}{1-z^2} & \frac{z^2+2\epsilon}{z(1-z^2)} \end{pmatrix}\vec{I}\,. \tag{17}$$

The boundary conditions can be fixed at $z = 1$

$$I_1 = I_2 = 1 + 2\epsilon + (1 + 2\zeta_2)\epsilon^2 + \mathcal{O}(\epsilon^3)\,, \tag{18a}$$

$$\begin{aligned} I_3 = I_4 = \frac{3}{8}\zeta_2 + \frac{1}{4} + \Big(&+\frac{21}{16}\zeta_3 - \frac{9}{4}\log 2\zeta_2 + \frac{9}{8}\zeta_2 - \frac{1}{2}\Big)\epsilon + \Big(-\frac{63}{16}\zeta_4 + 9\mathrm{Li}_4(\tfrac{1}{2}) + \frac{9}{2}\log 2^2\zeta_2 + \frac{3}{8}\log 2^4 \\ &+\frac{63}{16}\zeta_3 - \frac{27}{4}\log 2\zeta_2 + \frac{1}{2}\zeta_2 + 1\Big)\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{18b}$$

We will now try to derive a series expression around $z = 1$.

b) Show (18). *(for the adventurous)*

c) Consider first the homogenous subsystem for $\vec{I}' = (I_3, I_4)$ with $x = 1 - z$

$$\partial_x\vec{I}' = \begin{pmatrix} \frac{1-x}{(x-2)x} & -\frac{1}{(x-2)(x-1)^2x} \\ -\frac{(1-x)^2}{(x-2)x} & \frac{1-x}{(x-2)x} \end{pmatrix}\vec{I}'\,. \tag{19}$$

Construct $\tilde{M}$ and $\bar{M}$

$$\bar{M} = \begin{pmatrix} 1 & 0 \\ \frac{1-x}{(x-2)x} & -\frac{1}{(x-2)(x-1)^2x} \\ \frac{2(x^2-2x+2)}{(x-2)^2x^2} & \frac{2(3x^2-6x+2)}{(x-2)^2(x-1)^3x^2} \end{pmatrix} \tag{20}$$

d) Find a solution $\vec{c}$ and write down the second-order differential equation for I_3. Make a Frobenius ansatz. What values of r are allowed?

e) Build a series solution for $r = 0$.

f) Construct the first-order differential equation to find the other solution for

$$\vec{c}_2 = \left(\frac{4}{x}\frac{1-x}{2-x}, 1\right). \tag{21}$$

g) Build a series solution for $r = -2$ and write down the full homogenous solution for the original differential equation.

h) Now we can consider the inhomogeneity. Show that for the k-th order in ϵ, it becomes

$$\vec{\mathcal{I}}^{(k)} = \frac{1}{x(x-2)}\left[\begin{pmatrix} \frac{1}{1-x} & -\frac{1}{1-x} \\ -(1-x)^2 & 1 \end{pmatrix} \cdot \left(I_1^{(k)}, I_2^{(k)}\right) + \begin{pmatrix} \frac{4x^2-8x-2}{x-1} & -\frac{2}{(x-1)^2} \\ -2(x-1)^2 & -\frac{2}{x-1} \end{pmatrix} \cdot \left(I_3^{(k-1)}, I_4^{(k-1)}\right)\right]. \tag{22}$$

i) Now construct a solution up to $\mathcal{O}(\epsilon^2)$. Evaluate the result at $z = 0.5$.

SOLUTION: A Mathematica calculation of this problem is given on the website.

a) It is trivial to derive a differential equation for eg. m_2^2 using `reduze`. From there we write

$$T = (m_2^2)^{d-4}\epsilon^{-2}\mathrm{diag}\left(\frac{m_2^4}{m_1^4}, \frac{m_2^2}{m_1^2}, \frac{m_2^4}{m_1^4}, \frac{m_2}{m_1}\right) \tag{23}$$

and hence A_z.

b) It is clear that $I_1 = I_2$ and $I_3 = I_4$ from symmetry for $z = 1$. I_1 is just a tadpole and trivially just

$$I_1 = I_2\Big|_{z=1} = (1+\epsilon)^2\Gamma[1-\epsilon]^2\Gamma[1+\epsilon]^2\,. \tag{24}$$

For I_3 we calculate the graph polynomials

$$\mathcal{U} = x_1x_2 + x_3x_2 + x_1x_3\,, \tag{25}$$

$$\mathcal{F} = -(x_1+x_2)\mathcal{U} - (x_1+x_2)x_3^2\,. \tag{26}$$

Therefore the integral is

$$I_3 = (-1)^{2\epsilon}\Gamma(1-\epsilon)^2\Gamma(2+2\epsilon)\int \mathrm{d}\vec{x}\; x_1\frac{\mathcal{U}^{1+3\epsilon}}{\mathcal{F}^{2+2\epsilon}} \tag{27}$$

$$= \Gamma(1-\epsilon)^2\Gamma(2+2\epsilon)\int \mathrm{d}\vec{x}\; x_1\mathcal{U}^{1+3\epsilon}(x_1+x_2)^{-2-2\epsilon}(\mathcal{U}+x_3^2)^{-2-2\epsilon}\,. \tag{28}$$

Using the Mellin-Barnes theorem to split $\mathcal{U} + x_3^2$

$$I_3 = \Gamma(1-\epsilon)^2\int_{-i\infty}^{+i\infty}\mathrm{d}\sigma\int \mathrm{d}\vec{x}\Gamma(-\sigma)\Gamma(2+\sigma+2\epsilon)\; x_1x_3^{2\sigma}(x_1+x_2)^{-2-2\epsilon}\mathcal{U}^{-1-\sigma+\epsilon}\,. \tag{29}$$

We can now solve the x_3 and x_2 integrals

$$I_3 = \Gamma(1-\epsilon)^2\int_{-i\infty}^{+i\infty}\mathrm{d}\sigma\frac{\Gamma(-\sigma)\Gamma(2\sigma+1)\Gamma(-\epsilon-\sigma)\Gamma(\epsilon+\sigma+1)\Gamma(\epsilon+\sigma+2)\Gamma(2\epsilon+\sigma+2)}{\Gamma(-\epsilon+\sigma+1)\Gamma(2\epsilon+2\sigma+3)}\,. \tag{30}$$

This integral can be solved in terms of ${}_3F_2$ functions by considering the residues at $\sigma = n$ and $\sigma = n - \epsilon$. Alternatively, we can solve it numerically for $\sigma = 1/3 + i\,x$

$$
\begin{aligned}
I_3 = &+ 0.8668502750680849136771556874922594459571062129525494141508343360137528 \\
&+ 0.3628423366548035363008187167277704240129460044840854565583099812335021\epsilon \\
&+ 2.897979437584699485862761102760735004697735930030264157054679832034451\epsilon^2 \\
&+ \mathcal{O}(\epsilon^3)\,.
\end{aligned}
\tag{31}
$$

We have evaluated the integral to 70 digits which is required to perform the rational number fit.

c) It is trivial to find the homogenous matrix M as given in (19). From there, constructing $\bar{M}$ is just a matter of calculating deriviatives.

d) We find $p = 2$

$$
\vec{c} = \left(\frac{2}{x-2} - \frac{2}{x}, \frac{2}{x-2} + \frac{2}{x-1} + \frac{2}{x}, 1 \right). \tag{32}
$$

Making the ansatz and expanding to the first order, we find $r(1+r) = 0$. We hence choose $r = 0$.

e) It is easy to see that $a_i = 1$ for all i and hence $f_1 = 1/(1-x)$ is a solution.

f) We have

$$
\begin{aligned}
(c_2)_j &= \sum_{n=0}^{p-1-j} \binom{p-n}{1+j} c_{p-n+1} \frac{\partial^{p-n-j-1} f_1}{\partial x^{p-n-j-1}} \\
&= \left(\frac{6x^2 - 12x + 4}{(1-x)x(x^2-3x+2)} + \frac{2}{(1-x)^2}, \frac{1}{1-x} \right),
\end{aligned}
\tag{33}
$$

which is equivalent to the given $\vec{c}_2$.

g) The solution for $\vec{c}_2$ is slightly more complicated but can still be written in closed form

$$
\begin{aligned}
f_1' &= \frac{1}{x^2} + \frac{1}{x} + \frac{3}{4} + \frac{x}{2} + \frac{5x^2}{16} + \frac{3x^3}{16} + \frac{7x^4}{64} + \frac{x^5}{16} + \frac{9x^6}{256} + \frac{5x^7}{256} + \frac{11x^8}{1024} + \frac{3x^9}{512} + \mathcal{O}(x^{10}) \\
&= \sum_{i=0}^{\infty} \frac{i+1}{2^i} x^{i-2} = \frac{4}{x^2(x-2)^2}
\end{aligned}
\tag{34}
$$

The integrated f_1' can also be found easily

$$
\begin{aligned}
f_1 = f_0 \int \mathrm{d}x f_1' = f_0 \int \mathrm{d}x &= \frac{1}{1-x} \Big(-\frac{1}{x} + \log x + \sum_{i=1}^{\infty} \frac{2+i}{2i} \frac{x^i}{2^i} \Big) \\
&= -\frac{1}{x} + \log x - 1 + x\Big(\log x - \frac{1}{4}\Big) + x^2 \log x + x^3 \Big(\log(x) + \frac{5}{48}\Big) + x^4 \Big(\log x + \frac{29}{192}\Big) + \ldots
\end{aligned}
\tag{35}
$$

The solution matrix F can be found as

$$F = \begin{pmatrix} 1+x+x^2+x^3 & -\frac{1}{x}+\log(x)-1+x\big(\log x-\frac{1}{4}\big)+x^2\log x+x^3\big(\log x+\frac{5}{48}\big) \\ 1+O\big(x^4\big) & \frac{1}{x}+\log(x)-1+\frac{x}{4}-\frac{x^3}{48} \end{pmatrix} + \mathcal{O}(x^4)\,. \tag{36}$$

h) There are two contributions to $\mathcal{I}$

- the lower sector contributions from I_1 and I_2 that are governed by the slices of $(A_x)_{3..4,1..2}$. These can be read of from the ϵ^0 part of (17) after substituting $z \to 1-x$.
- the lower order contribution from I_3 and I_4 that are governed by the $\mathcal{O}(\epsilon)$ part of $(A_x)_{3..4,3..4}$.

i) We calculate

$$G^{(k)} = F \cdot \left(\int \mathrm{d}x F^{-1} \frac{1}{2}\Big(\vec{\mathcal{I}}^{(k)}, \vec{\mathcal{I}}^{(k)}\Big) + \mathrm{diag}(c_1^{(k)}, c_2^{(k)}) \right) \tag{37}$$

order-by-order and then obtain $I_3 = G_{11} + G_{12}$ and $I_4 = G21 + G_{22}$. We can then take the limit $x \to 0$ to match the integration constants. Since we have a series expansion, this is fairly trivial. For $k = 0$,

$$G^{(0)} = \begin{pmatrix} c_1^{(0)} & -\frac{c_2^{(0)}}{x} + c_2^{(0)}(\log x - 1) \\ c_1^{(0)} & \frac{c_2^{(0)}}{x} + c_2^{(0)}(\log x - 1) \end{pmatrix} + \mathcal{O}(x)\,. \tag{38}$$

Since the limit $x \to 0$ is finite, $c_2^{(0)} = 0$. Matching (18), we then find $c_1^{(0)} = \frac{1}{4} + \frac{\pi^2}{16}$.

Pluggin in numbers we find

$$\vec{I} = \begin{pmatrix} 1 \\ 1 \\ 1.3581 \\ 0.4669 \end{pmatrix} + \begin{pmatrix} -0.7725 \\ 0.6137 \\ 0.4189 \\ 0.3800 \end{pmatrix} \epsilon + \begin{pmatrix} 2.5883 \\ 2.4781 \\ 3.3107 \\ 1.1768 \end{pmatrix} \epsilon^2 + \begin{pmatrix} -1.1793 \\ 2.1104 \\ 1.9080 \\ 1.2225 \end{pmatrix} \epsilon^3 + \mathcal{O}(\epsilon^4) \tag{39}$$