AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Practical guide to loop integration
Exercise Sheet 5

Summer Term 2024
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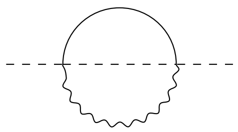
Issued: 14 May 2024

You might find the Mathematica packages `MB Tools` useful: `https://mbtools.hepforge.org/`. Consider using especially `MB.m`, `MBresolve.m`, and `barnesroutines.m`.

Exercise 1: Mellin Barnes expansion

In this example, we will consider once again a sunset diagram



$$= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2 - m^2\big]\big[k_2^2 - M^2\big]\big[(k_1 - k_2 - p)^2\big]} \tag{1}$$

with $p^2 = m^2$.

a) Use a single MB split to solve the Feynman integrals. There is no need to sum the MB series yet.

The MB will be of the form

$$I = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \Big(\frac{m^2}{M^2}\Big)^{-z} f(z) = \sum_{n=0}^{\infty} \Big(\frac{m^2}{M^2}\Big)^n f'(n)$$

with some $f(z)$ and $f'(n)$ assuming we have closed the contour on the correct side.

We can at this point decide that $0 < m \ll M$ and expand the integral. Conceptually, this is done by noting that the terms in the series of residues are suppressed by $(m^2/M^2)^n$. To expand to any order in m^2/M^2 we just truncate the series.

b) Calculate the integral up to $\mathcal{O}(m^4)$.

c) Solve the integral exactly in m by calculating the full series, expand in ϵ using `HypExp`. Finally expand in m to verify your result.

   (for the adventurous)

SOLUTION:

a) We have

$$I = -\Gamma(1-\epsilon)^2\Gamma(2\epsilon-1) \int \mathrm{d}\vec{x}\delta(\cdots) \frac{\Big(x_1x_2 + x_1x_3 + x_2x_3\Big)^{-3+3\epsilon}}{\Big(m^2x_1^2(x_2+x_3) + M^2x_2(x_1x_2+x_1x_3+x_2x_3)\Big)^{-1+2\epsilon}}\,.$$

We split at m^2

$$I = -(m^2)^{1-2\epsilon}\Gamma(1-\epsilon)^2 \int_{-i\infty}^{+i\infty} dz \left(\frac{M^2}{m^2}\right)^z \Gamma(-z)\Gamma(2\epsilon-1+z) \int dx_1 dx_2 dx_3 \delta(\cdots)$$
$$x_1^{2-2z-4\epsilon} x_2^z (x_2+x_3)^{1-z-2\epsilon}(x_1x_2+x_1x_3+x_2x_3)^{-3+3\epsilon+z}\,.$$

We can solve x_1 and x_2 and use the δ function for x_3

$$I = -(m^2)^{1-2\epsilon}\Gamma(1-\epsilon)^3 \int_{-i\infty}^{+i\infty} dz \left(\frac{m^2}{M^2}\right)^{-z}$$
$$\frac{\Gamma(-z)\Gamma(3-4\epsilon-2z)\Gamma(1-\epsilon-z)\Gamma(\epsilon+z)\Gamma(2\epsilon-1+z)}{\Gamma(3-3\epsilon-z)\Gamma(2-2\epsilon-z)}\,.$$

Closing the contour around the left poles we have

$$I = M^2\Gamma(1-\epsilon)^3\Bigg[\left(\frac{\mu^2}{mM}\right)^{2\epsilon}\sum_{n=0}^{\infty}\left(-\frac{m^2}{M^2}\right)^{n+1}\frac{\Gamma(2n-2\epsilon+3)\Gamma(-n+\epsilon-1)\Gamma(n+\epsilon)}{\Gamma(n-2\epsilon+3)\Gamma(n-\epsilon+2)}$$
$$+\left(\frac{\mu^2}{M^2}\right)^{2\epsilon}\sum_{n=0}^{\infty}\left(-\frac{m^2}{M^2}\right)^{n}\frac{\Gamma(2n+1)\Gamma(-n-\epsilon+1)\Gamma(n+\epsilon)\Gamma(n+2\epsilon-1)}{\Gamma(n+1)^2\Gamma(n-\epsilon+2)}\Bigg]\,.$$

b) Calculating the first few terms

$$I = \frac{1}{2}\left(\frac{\mu^2}{M^2}\right)^{2\epsilon}\Bigg[M^2\left(\frac{1}{\epsilon^2}+\frac{3}{\epsilon}+4\zeta_2+7\right)$$
$$+m^2\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{5}{2}-2\log\frac{m^2}{M^2}\right)+\frac{17}{4}-4\log\frac{m^2}{M^2}+\log^2\frac{m^2}{M^2}\right)+\mathcal{O}(m^4)\,.\Bigg]$$

c) We find

$$I = -M^{-4\epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(\epsilon)}{\Gamma(2-\epsilon)}\Bigg[M^2\Gamma(1-\epsilon)\Gamma(2\epsilon-1)\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\ 2\epsilon-1\\ 2-\epsilon\end{matrix};\frac{4m^2}{M^2}\right]$$
$$+m^2\left(\frac{m^2}{M^2}\right)^{\epsilon}\Gamma(\epsilon-1)\,{}_3F_2\left[\begin{matrix}1,\ \frac{3}{2}-\epsilon,\ \epsilon\\ 3-2\epsilon,\ 2-\epsilon\end{matrix};\frac{4m^2}{M^2}\right]\Bigg]$$

using `HypExp` we can expand this

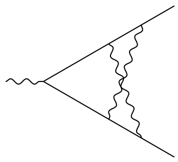
$$I = \frac{1}{2}m^2M^{-4\epsilon}\Bigg[\frac{1}{\epsilon^2}\left(3+\frac{1}{x}+x\right)+\frac{1}{\epsilon}\left(\frac{17}{2}+\frac{3}{x}+3x+4H_{-1}(x)-2H_0(x)\right)$$
$$+\frac{6}{x}+\frac{59}{4}+6x+\frac{4}{x}\zeta_2+8\zeta_2+4\zeta_2 x-\left(8+4x+x^2\right)H_0(x)$$
$$+\left(\frac{1}{x^2}+\frac{4}{x}+16+4x+x^2\right)H_{-1}(x)+4\left(\frac{1}{x}+x\right)H_{0,-1}(x)$$
$$-8\left(\frac{1}{x}+x\right)H_{-1,-1}(x)+4\left(\frac{1}{x}+x\right)H_{-1,0}(x)+2H_{0,0}(x)\Bigg]$$

where we have defined

$$x = \frac{1-\beta}{1+\beta}\quad \text{with}\quad \beta=\sqrt{1-\frac{4m^2}{M^2}}\,.$$

Exercise 2: Multiple Mellin Barnes

Consider the following non-planar integral



$$= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2\big]\big[k_2^2\big]\big[(k_1-p-q)^2\big]\big[(k_1-k_2)^2\big]\big[(k_1-k_2-q)^2\big]\big[(k_2-p)^2\big]}$$

with $p^2 = q^2 = 0$ and $(p+q)^2 = s$.

a) Solve the Feynman integration. This can be done using two Mellin Barnes splits

$$\frac{1}{(A_1+A_2+A_3)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z_1 \mathrm{d}z_2 A_1^{z_1}\Gamma(-z_1) A_2^{z_2}\Gamma(-z_2) A_3^{-\lambda-z_1-z_2}\Gamma(\lambda+z_1+z_2)\,.$$

Hint: You might find the substitution $x_2 \to x_6 x_2$ useful.

b) Resolve the singularities and expand in ϵ up to ϵ^0.

c) Use the Barnes Lemmas and PSLQ to solve the resulting integral

SOLUTION:

a) Setting $s=-1$, we have

$$I = \Gamma(1-\epsilon)^2\Gamma(2\epsilon+2)\int \mathrm{d}x_1\cdots\mathrm{d}x_6\delta(\cdots)\frac{\Big[(x_4+x_5)(x_2+x_6)+(x_1+x_3)(x_2+x_4+x_5+x_6)\Big]^{3\epsilon}}{\Big[x_2x_3x_4+x_1x_5x_6+x_1x_3(x_2+x_4+x_5+x_6)\Big]^{2(\epsilon+1)}}$$

We use MB for the three terms of the denominator. Afterwards we can trivially solve x_1 and x_3 and arrive at

$$I = \frac{\Gamma(1-\epsilon)^2}{\Gamma(-3\epsilon)}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z_1\mathrm{d}z_2\int \mathrm{d}x_2\mathrm{d}x_4\mathrm{d}x_5\mathrm{d}x_6\delta(\cdots)\frac{x_2^{z_1}x_4^{z_1}x_5^{z_2}x_6^{z_2}(x_2+x_4+x_5+x_6)^{2\epsilon}}{(x_4+x_5)^{2+z_1+z_2+\epsilon}(x_2+x_6)^{2+z_1+z_2+\epsilon}}$$
$$\times\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_1-2\epsilon-1)\Gamma(-z_2-2\epsilon-1)\Gamma(z_1+z_2+\epsilon+2)\Gamma(z_1+z_2+2\epsilon+2)\,.$$

By substituting $x_2\to x_6x_2$, we can turn $x_2^\alpha x_6^\beta(x_2+x_6)^\gamma$ into $x_2^\alpha x_6^{1+\alpha+\beta+\gamma}(1+x_2)^\gamma$. Afterwards, we can solve x_6, x_2, and x_4. Finally, we use the δ function for x_5

$$I = \frac{\Gamma(1-\epsilon)^2\Gamma(-\epsilon)^2}{\Gamma(-3\epsilon)\Gamma(-2\epsilon)}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z_1\mathrm{d}z_2 f(z_1,z_2)$$

where we have defined

$$f(z_1,z_2) = \frac{\Gamma(\epsilon+z_1+z_2+2)\Gamma(2\epsilon+z_1+z_2+2)}{\Gamma(z_1+z_2+2)^2}$$
$$\Gamma(-z_1)\Gamma(z_1+1)^2\Gamma(-2\epsilon-z_1-1)\Gamma(-z_2)\Gamma(z_2+1)^2\Gamma(-2\epsilon-z_2-1)$$

b) After resolving the poles we are left with

$$I = +\underbrace{\operatorname*{res}_{z_2=-1-2\epsilon}\Big[\operatorname*{res}_{z_1=-1-2\epsilon} f(z_1,z_2)\Big]}_{I^{(0)}} + \underbrace{\int_{-1/3-\mathrm{i}\infty}^{-1/3+\mathrm{i}\infty}\mathrm{d}z_1\int_{-3/2-\mathrm{i}\infty}^{-3/2+\mathrm{i}\infty}\mathrm{d}z_2 f(z_1,z_2)}_{I^{(2)}}$$
$$\underbrace{+\int_{-2/3-\mathrm{i}\infty}^{-2/3+\mathrm{i}\infty}\mathrm{d}z_1\operatorname*{res}_{z_2=-1-2\epsilon}f(z_1,z_2) - \int_{-2/3-\mathrm{i}\infty}^{-2/3+\mathrm{i}\infty}\mathrm{d}z_2\operatorname*{res}_{z_1=-1-2\epsilon}f(z_1,z_2)}_{I^{(1)}}\,.$$

We can now expand in ϵ. $I^{(0)}$ and $I^{(1)}$ are divergent with ϵ^{-4} and ϵ^{-2} poles respectively.

c) Let us focus on the finite double integral $I^{(2)}$ first

$$I^{(2)} = \underbrace{\left[\int_{-1/3-\mathrm{i}\infty}^{-1/3+\mathrm{i}\infty} \mathrm{d}z_1 \Gamma(-z_1)\Gamma(z_1+1)^2\Gamma(-z_1-1)\right]}_{-\zeta_2} \underbrace{\left[\int_{-3/2-\mathrm{i}\infty}^{-3/2+\mathrm{i}\infty} \mathrm{d}z_2 \Gamma(-z_2)\Gamma(z_2+1)^2\Gamma(-z_2-1)\right]}_{+\zeta_2}.$$

The sign difference comes from the slightly different contour. The single integrals can not completely be solved with the Barnes Lemma though some headway can be made. We find for the poles

$$I^{(1)} = -\frac{\zeta_2}{2\epsilon^2} - \frac{\gamma_E\zeta_2 - 3\zeta_3}{2\epsilon} + \int_{-2/3-\mathrm{i}\infty}^{-2/3+\mathrm{i}\infty} \mathrm{d}z f'(z)\,.$$

f' is a complicated function involving ψ^2 and ψ'. We could further attempt the Barnes Lemmas. However, out of sheer lazyness we just PSLQ the entire thing using the basis $\{\gamma_E^2\zeta_2, \gamma_E\zeta_3, \zeta_4\}$

$$\int_{-2/3-\mathrm{i}\infty}^{-2/3+\mathrm{i}\infty} \mathrm{d}z f'(z) = 4.82717690481472 = -\frac{1}{4}\zeta_2\gamma_E^2 + \frac{3}{2}\gamma_E\zeta_3 + \frac{29}{8}\zeta_4$$

Finally,

$$I = \frac{1}{\epsilon^4} - \frac{5\zeta_2}{\epsilon^2} - \frac{27\zeta_3}{\epsilon} - \frac{115\zeta_4}{2}\,.$$

Had we used PSLQ earlier and included the pre-factor the basis would have been a lot easier. Without much effort we could have calculated

$$I = \frac{1.}{\epsilon^4} + \frac{0.}{\epsilon^3} - \frac{-8.224675}{\epsilon^2} - \frac{-32.4555}{\epsilon} - 62.2336$$

and just guessed the answer.