AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Practical guide to loop integration
Exercise Sheet 4

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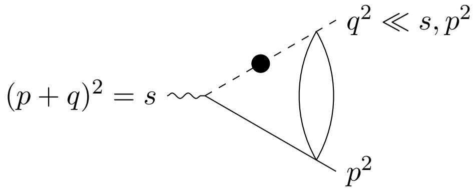
Exercise 1: top quark decay

Consider the following integral that appears in the two-loop calculation of the top decay



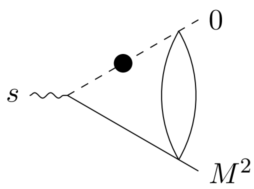
(1)

The dashed line corresponds to light b quark and the solid line to the heavy t quark.

a) Find the momentum regions that contribute to this process.

b) Show that the hard region does in fact equal the same calculation with massless b quarks, i.e. $m = 0$.

c) The result for $m = 0$ can be found in the literature as



$$
= \frac{(M^2)^{-2\epsilon}}{M^2 - s} \Bigg[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \frac{1+y}{y} H_1(y) + 4 + \frac{2(1+y)}{y} \Big(H_1(y) + H_{1,1}(y) \Big) + \frac{2}{y} \frac{1+y}{1-y} \Big(H_{0,1}(y) - y\zeta_2 \Big) \Bigg] . \tag{2}
$$

Show by explicit calculation of the remaining regions that the complete integral has no $1/\epsilon^2$ pole but a $\log(m^2/M^2)$.

SOLUTION: We write the integral as

$$
I = \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2 + 2k_1 \cdot p\big] \big[k_1^2 - 2k_1 \cdot q\big]^2 \big[k_2^2 - M^2\big] \big[(k_1 - k_2)^2 - M^2\big]}
$$

with $p^2 = M^2$ and $q^2 = m^2 \sim \lambda^2$.

a) We have two loop momenta that can scale differently. Hence, a region is fully specified by six numbers

$$
k_1 \sim (\lambda^a, \lambda^b, \lambda^c) \quad \text{and} \quad k_2 \sim (\lambda^d, \lambda^e, \lambda^f) .
$$

We obviously have a hard region $k_1 \sim k_2 \sim (1,1,1)$. We further have a collinear-hard region $k_1 \sim (\lambda^2, 1, \lambda)$ and $k_2 \sim (1,1,1)$. All other regions are scaleless. For example the collinear-collinear region $k_1 \sim k_2 \sim (\lambda^2, 1, \lambda)$

$$I_{cc} = \frac{\lambda^4}{M^4} \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[2k_1 \cdot p_+\big]\big[k_1^2 - 2k_1 \cdot q\big]} = 0$$

b) We have with $q = q_- + q_\perp$ with $q_- \sim 1$ and $q_\perp \sim \lambda$

$$\begin{aligned} I_{hh} &= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2 + 2k_1 \cdot p\big]\big[k_1^2 - 2k_1 \cdot (q_- + \lambda q_\perp)\big]^2 \big[k_2^2 - M^2\big]\big[(k_1 - k_2)^2 - M^2\big]} \\ &= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2 + 2k_1 \cdot p\big]\big[k_1^2 - 2k_1 \cdot q_-\big]^2 \big[k_2^2 - M^2\big]\big[(k_1 - k_2)^2 - M^2\big]} + \mathcal{O}(\lambda)\,. \end{aligned}$$

Since $q_-^2 = 0$, we now have the massless result evaluated at q_- instead of q.

c) The other region, I_{ch}, is

$$I_{ch} = \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[2k_{1-} \cdot p_+\big]\big[k_1^2 - 2(q_- \cdot k_{1+} + q_\perp \cdot k_{1\perp})\big]^2 \big[k_2^2 - M^2\big]\big[k_2^2 - 2k_{1-} \cdot k_{2+} - M^2\big]} + \mathcal{O}(\lambda)\,.$$

In the scalar products between q and k_1 we can just take the full k_1 because the q will project out the necessary component. The $k_{1-} \cdot k_{2+}$ is, however, a problem.

$$I_{ch} = \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[2k_1 \cdot p_+\big]\big[k_1^2 - 2k_1 \cdot (q_- + q_\perp)\big]^2 \big[k_2^2 - M^2\big]\big[k_2^2 - 2k_{1-} \cdot k_{2+} - M^2\big]} + \mathcal{O}(\lambda)\,.$$

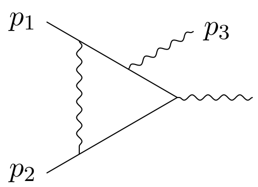
By solving first the k_2 integration, we write $k_{1-} \cdot k_{2+} = k_{1-} \cdot k_2$

$$\begin{aligned} I_{ch} &= (M^2)^{-\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon) \int \mathrm{d}x_1 \mathrm{d}x_2 \frac{\delta(\cdots)}{(x_1+x_2)^2} \int [\mathrm{d}k_1] \frac{1}{\big[2k_1 \cdot p_+\big]\big[k_1^2 - 2k_1 \cdot (q_- + q_\perp)\big]^2} + \mathcal{O}(\lambda) \\ &= 2^\epsilon (M^2)^{-\epsilon}\Gamma(1-\epsilon)^2\Gamma(\epsilon)\Gamma(1+\epsilon) \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_1 \mathrm{d}y_2 \frac{\delta(\cdots)\delta(\cdots)}{(x_1+x_2)^2} \\ &\qquad y_1^{-1+\epsilon}\big(2m^2 y_1 + (m^2 + M^2 - s)y_2\big)^{-1-\epsilon} \\ &= -(M^2)^{-2\epsilon}\Big(\frac{m^2}{M^2}\Big)^{-\epsilon} \frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)^2}{m^2 + M^2 - s} = -(M^2)^{-2\epsilon}\Big(\frac{m^2}{M^2}\Big)^{-\epsilon} \frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)^2}{M^2 - s} + \mathcal{O}(\lambda) \\ &= \frac{(M^2)^{2\epsilon}}{M^2 - s}\Bigg[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\log\frac{m^2}{M^2} + \mathcal{O}(\epsilon^0)\Bigg] + \mathcal{O}(\lambda)\,. \end{aligned}$$

The $1/\epsilon^2$ pole cancels exactly the one in (2)

Exercise 2: Soft approximation

Consider the following diagram



$$Q = \int [\mathrm{d}k] \frac{1}{\big[k^2\big]\big[(k-p_1)^2 - m^2\big]\big[(k-p_1+p_3)^2 - m^2\big]\big[(k+p_2)^2 - m^2\big]} \tag{3}$$

with the energy of the photon soft compared to $s = Q^2$ and $p_1^2 = p_2^2 = m^2 \ll s$. This can be phrased in a Lorentz invariant way by requiring that all invariants

$$\sigma_{i3} = 2p_i \cdot p_3 \sim m \sim \lambda\,. \tag{4}$$

a) Use the method of regions in the parametric representation to find all six regions that contribute to this integral.

b) Consider the region $\vec{r}^{(1)} = (0, -1, -1, 1)$, i.e. $\mathcal{P}_1 \sim 1$, $\mathcal{P}_2 \sim \lambda^{-1}$, $\mathcal{P}_3 \sim \lambda^{-1}$, $\mathcal{P}_4 \sim \lambda$. Show that this integral is not finite in dimensional regularisation.

Such behaviour is not uncommon and usually points to a broken symmetry eg. in SCET. It is usually addressed by using analytic regularisation, i.e.

$$\begin{aligned} &\int [\mathrm{d}k] \frac{1}{\big[k^2\big]\big[(k-p_1)^2 - m^2\big]\big[(k-p_1+p_3)^2 - m^2\big]\big[(k+p_2)^2 - m^2\big]} \\ &\to \big(-\nu^2\big)^\eta \int [\mathrm{d}k] \frac{1}{\big[k^2\big]\big[(k-p_1)^2 - m^2\big]\big[(k-p_1+p_3)^2 - m^2\big]^{1+\eta}\big[(k+p_2)^2 - m^2\big]}\,. \end{aligned} \tag{5}$$

We have introduced an additional regulator η that we will take to zero as soon as we have added all regions. Crucially, $\eta \to 0$ needs to be done *before* $\epsilon \to 0$.

c) Calculate the integral, up to $\mathcal{O}(\lambda^0)$. Add all regions and set $\eta \to 0$ and finally $\epsilon \to 0$.

SOLUTION: We have

$$\begin{aligned} I = \Gamma(1-\epsilon)\Gamma(2+\epsilon) \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \delta(\cdots)\big(x_1+x_2+x_3+x_4\big)^{2\epsilon}\Big[-s(x_2+x_3)x_4 \\ + m^2(x_2+x_3+x_4)^2 + \sigma_{13}(x_1x_3 + 2x_4x_3 + x_2x_4) + \sigma_{23}(x_2+2x_3)x_4\Big]^{-\epsilon-2} \end{aligned}$$

a) Using `asy` and `QHull`

```
Get["/path/to/asy2.1.m"];
SetOptions[QHull, Executable -> "/path/to/qhull"]
props = {
  k1^2,
  (k1 - p1)^2 - m^2,
```

```
    (k1 - p1 + p3)^2 - m^2,
    (k1 + p2)^2 - m^2
  };
  onshell = {
    p1^2 -> m^2, p2^2 -> m^2, p3^2 -> 0,
    p1 p2 -> 1/2 (-2 m^2 + s - σ13 - σ23),
    p1 p3 -> σ13/2,
    p2 p3 -> σ23/2,
  };

  AlphaRepExpand[{k1},
    props,
    onshell,
    {σ13 -> x, σ23 -> x, m -> x, s -> 1}
  ]
  (* Asy2.1 *)
  (* Variables for UF: {k1, p1, p2, p3} *)
  (* {{0, -1, -1, 1}, {0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 2},
   ↪ {0, 2, 2, 0}, {0, 3, 3, 1}} *)
```

we find

$$
\begin{aligned}
\vec{r}^{(1)} &= (0,-1,-1,1)\,,\\
\vec{r}^{(2)} &= (0,0,0,0)\,,\\
\vec{r}^{(3)} &= (0,0,0,1)\,,\\
\vec{r}^{(4)} &= (0,0,1,2)\,,\\
\vec{r}^{(5)} &= (0,2,2,0)\,,\\
\vec{r}^{(6)} &= (0,3,3,1)\,.
\end{aligned}
$$

b) In $\vec{r}^{(1)}$, we have $x_1 \sim 1$, $x_2 \sim x_3 \sim \lambda^{-1}$, $x_4 \sim m \sim \sigma_{13} \sim \sigma_{23} \sim \lambda$. Hence up to a global pre-factor of λ

$$
\begin{aligned}
I_1 &\propto \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)\big(x_2+x_3\big)^{2\epsilon}\Big[-s(x_2+x_3)x_4+m^2(x_2+x_3)^2+\sigma_{13}x_1x_3\Big]^{-\epsilon-2}\\
&\propto \int \mathrm{d}x_2\mathrm{d}x_3\delta(\cdots)x_3^{-1}(x_2+x_3)^{-1}\,.
\end{aligned}
$$

This integral integrates to $\Gamma(0)$ which is not finite.

c) We find after setting the power of the third propagator to $1+\eta$

$$
\begin{aligned}
I_1 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \\
&\quad \int \mathrm{d}\vec{x}\, x_3^\eta (x_2+x_3)^{\eta+2\epsilon} \Big[m^2 (x_2+x_3)^2 - s x_4 (x_2+x_3) + \sigma_{13} x_1 x_3 \Big]^{-\eta-\epsilon-2} \\
I_2 &= (-\nu^2)^\eta (-s)^{-\eta-\epsilon-2} \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \\
&\quad \int \mathrm{d}\vec{x}\, x_3^\eta x_4^{-\eta-\epsilon-2} (x_2+x_3)^{-\eta-\epsilon-2} (x_1+x_2+x_3+x_4)^{\eta+2\epsilon} \\
I_3 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \int \mathrm{d}\vec{x}\, x_3^\eta (x_1+x_2+x_3)^{\eta+2\epsilon} \Big[\sigma_{13} x_1 x_3 - s(x_2+x_3) x_4 \Big]^{-\eta-\epsilon} \\
I_4 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \int \mathrm{d}\vec{x}\, x_3^\eta (x_1+x_2)^{\eta+2\epsilon} \Big[m^2 x_2^2 - s x_4 x_2 + \sigma_{13} x_1 x_3 \Big]^{-\eta-\epsilon-2} \\
I_5 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \int \mathrm{d}\vec{x}\, x_3^\eta x_4^{-\eta-\epsilon-2} (x_1+x_4)^{\eta+2\epsilon} \Big[-s(x_2+x_3) + m^2 x_4 \Big]^{-\eta-\epsilon-2} \\
I_6 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \int \mathrm{d}\vec{x}\, x_3^\eta x_1^{\eta+2\epsilon} \Big[m^2 x_4^2 - s(x_2+x_3) x_4 + \sigma_{13} x_1 x_3 \Big]^{-\eta-\epsilon-2}
\end{aligned}
$$

All of these integrals can be trivially solved

$$
\begin{aligned}
I_1 &= \frac{1}{(-s)\sigma_{13}} \Big(\frac{-\nu^2}{m^2}\Big)^\eta \Big(\frac{\mu^2}{m^2}\Big)^\epsilon \frac{\Gamma(\eta)\Gamma(1-\epsilon)\Gamma(\epsilon+\eta)}{\Gamma(\eta+1)^2} \\
I_2 &= \frac{1}{(-s)(-s)} \Big(\frac{-\nu^2}{-s}\Big)^\eta \Big(\frac{\mu^2}{-s}\Big)^\epsilon \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-\epsilon-\eta-1)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+2)\Gamma(-2\epsilon-\eta)} \\
I_3 &= \frac{1}{(-s)\sigma_{13}} \Big(\frac{-\nu^2}{\sigma_{13}}\Big)^\eta \Big(\frac{\mu^2}{\sigma_{13}}\Big)^\epsilon \frac{\Gamma(-\epsilon)^2\Gamma(-\epsilon-\eta)\Gamma(\epsilon+\eta+1)}{\Gamma(\eta+1)\Gamma(-2\epsilon-\eta)} \\
I_4 &= \frac{1}{(-s)\sigma_{13}} \Big(\frac{-\nu^2}{\sigma_{13}}\Big)^\eta \Big(\frac{\mu^2}{m^2}\Big)^\epsilon \frac{\Gamma(-\eta)\Gamma(1-\epsilon)\Gamma(-2\epsilon)\Gamma(\epsilon)}{\Gamma(-2\epsilon-\eta)} \\
I_5 &= \frac{1}{(-s)(-s)} \Big(\frac{-\nu^2}{-s}\Big)^\eta \Big(\frac{\mu^2}{m^2}\Big)^\epsilon \frac{\Gamma(1-\epsilon)\Gamma(\epsilon)\Gamma(-2\epsilon-\eta-1)}{\Gamma(-2\epsilon-\eta)} \\
I_6 &= \frac{1}{(-s)\sigma_{13}} \Big(\frac{-\nu^2}{\sigma_{13}}\Big)^\eta \Big(\frac{\mu^2}{m^2}\frac{s^2}{\sigma_{13}^2}\Big)^\epsilon \frac{\Gamma(1-\epsilon)\Gamma(-2\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon+\eta+1)}{\Gamma(\eta+1)}
\end{aligned}
$$

Clearly, I_1, I_3, I_4, and I_6 start at λ^{-1} and need to be expanded one order higher. This results in

$$
\begin{aligned}
I &= \frac{1}{\lambda}\frac{1}{-s\sigma_{13}} \bigg(\frac{1}{\epsilon}\log\frac{m^2}{-s} + \log\frac{m^2}{-s}\log\frac{-s\mu^2}{\sigma_{13}^2} - \zeta_2 \bigg) \\
&\quad - \frac{\sigma_{13}+\sigma_{23}}{\sigma_{13}}\frac{1}{s^2} \bigg(\frac{1}{\epsilon}\Big(\log\frac{m^2}{-s}+1\Big) + \log\frac{m^2}{-s}\Big(\log\frac{-s\mu^2}{\sigma_{13}^2} - 4\Big) - \zeta_2 + 2\log\frac{m\mu}{\sigma_{13}} - 2 \bigg)
\end{aligned}
$$