AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Practical guide to loop integration
Exercise Sheet 3

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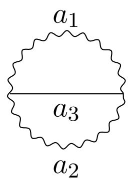
Exercise 1: Vacuum sunset

Calculate the vacuum sunset integral

$$I(a_1, a_2, a_3) =$$



$$= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2\big]^{a_1}\big[k_2^2\big]^{a_2}\big[(k_1-k_2)^2-m^2\big]^{a_3}} \tag{1}$$

for the cases $I(1,2,1)$ and $I(1,1,2)$ Why is this an easy integral to do?

SOLUTION: We can easily cut-construct the integral. The $\mathcal{U}$ term is all 2-cuts, i.e. $\mathcal{U} = x_1x_2 + x_2x_3 + x_1x_3$. The this is a vacuum integral, the momentum transfer across the 3-cut vanishes and we are left with $\mathcal{F} = m^2\ x_3\, \mathcal{U}$. Hence,

$$\begin{aligned} I(a_1,a_2,a_3) = \Gamma(1-\epsilon)^2(-1)^{a_1+a_2+a_3}(m^2)^{-a_1-a_2-a_3+d}\frac{\Gamma(-d+a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \\ \times \int \mathrm{d}\vec{x} x_1^{a_1-1}x_2^{a_2-1}x_3^{-a_1-a_2+d-1}\mathcal{U}^{-d/2}\,. \end{aligned} \tag{2}$$

We can calculate the integral trivially

$$\int \mathrm{d}\vec{x} x_1^{a_1-1}x_2^{a_2-1}x_3^{-a_1-a_2+d-1}\mathcal{U}^{-d/2} = \frac{\Gamma\big(\frac{d}{2}-a_1\big)\Gamma\big(\frac{d}{2}-a_2\big)\Gamma\big(-\frac{d}{2}+a_1+a_2\big)}{\Gamma\big(\frac{d}{2}\big)}\,. \tag{3}$$

Hence,

$$I(1,2,1) = (m^2)^{-2\epsilon}\frac{\Gamma(1-\epsilon)^3\Gamma(-\epsilon)\Gamma(2\epsilon)\Gamma(\epsilon+1)}{\Gamma(2-\epsilon)}\,, \tag{4}$$

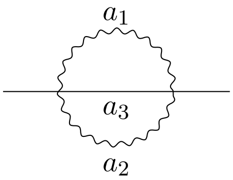
$$I(1,1,2) = (m^2)^{-2\epsilon}\frac{\Gamma(1-\epsilon)^4\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(2-\epsilon)}\,. \tag{5}$$

This integral is very simple because it only has one scale, m^2. Since we could find the exponent of m^2 by dimensional analysis, the integral really is a function of ϵ and nothing else. The fact that it is a vacuum integral further simplifies matters as $\mathcal{F} \propto \mathcal{U}$.

Exercise 2: Off-shell sunset

Now consider the integral with an external leg

$$I(a_1,a_2,a_3) = $$



$$= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\left[k_1^2\right]^{a_1}\left[k_2^2\right]^{a_2}\left[(k_1-k_2-p)^2-m^2\right]^{a_3}}\,, \tag{6}$$

with $p^2 = s = x\ m^2$.

a) Why can we not use the same calculation from the previous problem?

b) Show that this problem is solved by using the master integrals $\vec{I} = (I(1,2,1), I(1,1,2))^T$. These integrals are both starting at ϵ^{-2}.

c) Derive the differential equation matrices and show that

$$A_s = \begin{pmatrix} -\frac{(d-3)m^2+(d-4)s}{s(m^2-s)} & -\frac{(d-3)m^2}{s(m^2-s)} \\ \frac{d-4}{2s} & \frac{d-4}{2s} \end{pmatrix} \quad \text{and} \quad A_{m^2} = \begin{pmatrix} \frac{2d-7}{m^2-s} & \frac{d-3}{m^2-s} \\ -\frac{d-4}{2m^2} & \frac{d-4}{2m^2} \end{pmatrix}\,. \tag{7}$$

d) Find the mass scaling of each integral. Then rescale each integral such that it is finite and has vanishing mass scaling. Is the resulting system canonical?

e) Show that the matrix T achieves a canonical basis.

$$T = \frac{(m^2)^{-2\epsilon}}{\epsilon^2\ x}\left[\begin{pmatrix} -1 & 1-x \\ x & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 2 & -2(1-x) \\ -1-3x & 1-x \end{pmatrix}\right]\,. \tag{8}$$

f) Show that the boundary conditions $\vec{J_0}$ of the canonical system $\vec{J} = T^{-1}\vec{I}$ are

$$J_{0,1} = J_{0,2} = \frac{\epsilon\Gamma(1-\epsilon)^3\Gamma(\epsilon+1)\Gamma(2\epsilon-1)}{3\epsilon-1}\,. \tag{9}$$

g) Solve the remaining system up to ϵ^0 in $\vec{I}$.

SOLUTION:

a) We now have a more complicated $\mathcal{F}$ which is no longer just $\mathcal{F} \propto \mathcal{U}$

b) We perform the same IBP reduction we did in Sheet 2 with the family completed as

$$I(a_1,a_2,a_3,a_4,a_5) = \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\left[k_1^2\right]^{a_1}\left[k_2^2\right]^{a_2}\left[(k_1-k_2-p)^2-m^2\right]^{a_3}\left[k_1\cdot p\right]^{a_4}\left[k_2\cdot p\right]^{a_5}}\,. \tag{10}$$

c) For ∂_s we simplify have

$$\frac{\partial}{\partial s} I(a_1, a_2, a_3, 0, 0) = \frac{1}{2s} \int [\mathrm{d}k_1][\mathrm{d}k_2] p \cdot \frac{\partial}{\partial p} \frac{1}{\left[k_1^2\right]^{a_1} \left[k_2^2\right]^{a_2} \left[(k_1 - k_2 - p)^2 - m^2\right]^{a_3}} \tag{11}$$

$$\begin{aligned} = \frac{a_3}{s} \Bigg(& \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{2k_1 \cdot p}{\left[k_1^2\right]^{a_1} \left[k_2^2\right]^{a_2} \left[(k_1 - k_2 - p)^2 - m^2\right]^{a_3+1} \left[k_1 \cdot p\right]^{-1}} \\ & - \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{k_2 \cdot p}{\left[k_1^2\right]^{a_1} \left[k_2^2\right]^{a_2} \left[(k_1 - k_2 - p)^2 - m^2\right]^{a_3+1} \left[k_2 \cdot p\right]^{-1}} \\ & - s \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\left[k_1^2\right]^{a_1} \left[k_2^2\right]^{a_2} \left[(k_1 - k_2 - p)^2 - m^2\right]^{a_3+1}} \Bigg) . \end{aligned} \tag{12}$$

Since m only appears internally, ∂_{m^2} cannot be written in terms of ∂_p, instead we have

$$\frac{\partial}{\partial m^2} I(a_1, a_2, a_3) = a_3 \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\left[k_1^2\right]^{a_1} \left[k_2^2\right]^{a_2} \left[(k_1 - k_2 - p)^2 - m^2\right]^{a_3+1}} . \tag{13}$$

We can re-write this in terms of raising and lowering operators

$$\partial_s = \frac{a_3}{s} \mathbf{3}^+ \mathbf{4}^- - \frac{a_3}{s} \mathbf{3}^+ \mathbf{5}^- - a_3 \mathbf{3}^+ , \tag{14}$$

$$\partial_{m^2} = a_3 \mathbf{3}^+ . \tag{15}$$

Applying this to our master integrals and performing the reduction yields

$$\frac{\partial I_1}{\partial s} = -\frac{dm^2 + ds - 3m^2 - 4s}{s(m^2 - s)} I_1 - m^2 \frac{d-3}{s(m^2 - s)} I_2 \tag{16}$$

$$\frac{\partial I_2}{\partial m^2} = \frac{2(d-7)}{m^2 - s} I_1 + \frac{d-3}{m^2 - s} I_2 \tag{17}$$

$$\frac{\partial I_1}{\partial s} = \frac{d-4}{2s} I_1 + \frac{d-4}{2s} I_2 \tag{18}$$

$$\frac{\partial I_2}{\partial m^2} = -\frac{d-4}{2m^2} I_1 + \frac{d-4}{2m^2} I_2 . \tag{19}$$

From this we can read of A_s and A_{m^2}.

d) Calculating $sA_s + m^2 A_{m^2}$ we have $\mathrm{diag}(-4 + d, -4 + d)$. Since both integrals start at ϵ^{-2}, we need to multiply with $\epsilon^2 (m^2)^{4-d}$. Choosing $T = \epsilon^{-2} (m^2)^{d-4}$ achieves this aim. The new matrices B_s and B_{m^2} are

$$B_s = \begin{pmatrix} \frac{m^2(2\epsilon-1)+2s\epsilon}{s(m^2-s)} & \frac{m^2(2\epsilon-1)}{s(m^2-s)} \\ -\frac{\epsilon}{s} & -\frac{\epsilon}{s} \end{pmatrix} \quad \text{and} \quad B_{m^2} = \begin{pmatrix} \frac{-2m^2\epsilon+m^2-2s\epsilon}{m^4-m^2 s} & \frac{1-2\epsilon}{m^2-s} \\ \frac{\epsilon}{m^2} & \frac{\epsilon}{m^2} \end{pmatrix} . \tag{20}$$

Setting $x = s/m^2$

$$B_x = \begin{pmatrix} \frac{1-2(x+1)\epsilon}{(x-1)x} & \frac{1-2\epsilon}{(x-1)x} \\ -\frac{\epsilon}{x} & -\frac{\epsilon}{x} \end{pmatrix} \tag{21}$$

is pre-canonical.

e) We write

$$B_x = T^{-1}\Big(A_s \frac{\partial s}{\partial x}\Big) T - T^{-1}\frac{\partial T}{\partial x} = \epsilon \left[\frac{1}{\eta_1}\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} + \frac{1}{\eta_2}\begin{pmatrix} 0 & 0 \\ 2 & -4 \end{pmatrix} \right], \tag{22}$$

with $\eta_1 = x$ and $\eta_2 = x - 1$.

f) Consider T^{-1} for $x = 0$

$$T^{-1}\Big|_{x=0} = (m^2)^{2\epsilon}\epsilon^2 \begin{pmatrix} -\frac{\epsilon}{6\epsilon^2-5\epsilon+1} & \frac{1}{1-3\epsilon} \\ -\frac{\epsilon}{6\epsilon^2-5\epsilon+1} & \frac{1}{1-3\epsilon} \end{pmatrix}. \tag{23}$$

Multiplying with $\vec{I}_0$ from the previous problem gives two times the same result

$$J_{0,1} = J_{0,2} = \frac{\epsilon^2}{1-3\epsilon}\frac{\Gamma(1-\epsilon)^4\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(2-\epsilon)} - \frac{\epsilon^3}{6\epsilon^2-5\epsilon+1}\frac{\Gamma(1-\epsilon)^3\Gamma(-\epsilon)\Gamma(2\epsilon)\Gamma(\epsilon+1)}{\Gamma(2-\epsilon)}, \tag{24}$$

which can easily be simplified to the required expression.

g) By using the definition of the G function, this is fairly simple

$$\vec{J}^{(0)} = \Big(\frac{1}{2}, \frac{1}{2}\Big), \tag{25}$$

$$\vec{J}^{(1)} = \Big(\frac{5}{2}, \frac{5}{2} - G(1,x)\Big), \tag{26}$$

$$\vec{J}^{(2)} = \Big(\frac{19}{2} + 2\zeta_2 - G(0,1,x), \frac{19}{2} - 5G(1,x) + 2\zeta_2 - 2G(0,1,x) + 4G(1,1,x)\Big). \tag{27}$$

And hence for $\vec{I} = T\vec{J}$

$$(m^2)^{2\epsilon}\vec{I} = \Big\{ -\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\Big(-\frac{3}{2} + \frac{x-1}{x}G(1,x)\Big) - \frac{9}{2} - 2\zeta_2 + 3\frac{x-1}{x}G(1,x) + \frac{-1+2x}{x}G(0,1,x) - 4\frac{x-1}{x}G(1,1,x) + \mathcal{O}(\epsilon), \tag{28}$$

$$\frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} - \frac{1}{2} + 2\zeta_2 + \frac{x-1}{x}G(1,x) - G(0,1,x) + \mathcal{O}(\epsilon) \Big\}. \tag{29}$$