Practical guide to loop integration
Exercise Sheet 2

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

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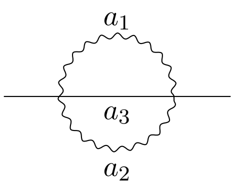
Exercise 1: Sunset diagram   Consider the following integral

$$I(a_1, a_2, a_3; \mathcal{N}) = \mathcal{N} \times$$



$$= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{\mathcal{N}}{\left[k_1^2\right]^{a_1} \left[k_2^2\right]^{a_2} \left[(k_1 - k_2 - p)^2 - m^2\right]^{a_3}} \tag{1}$$

with $p^2 = m^2 \neq 0$ and an arbitrary numerator $\mathcal{N}$.

a) Find a complete family. We know that it will need $\ell(1 + \ell + 2\rho)/2 = 5$ propagators.

b) Identify the sectors in which all integrals vanish.

c) Consider the IBP generated through $\partial_{k_1^\mu}(k_1^\mu I)$. Use it to show that

$$\int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{k_2 \cdot p}{\left[k_1^2\right] \left[k_2^2\right] \left[(k_1 - k_2 - p)^2 - m^2\right]^2} = \frac{3-d}{2} \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\left[k_1^2\right] \left[k_2^2\right] \left[(k_1 - k_2 - p)^2 - m^2\right]} \tag{2}$$

d) Now find all six seed identities as a function of $a_1, \ldots, a_5$.

e) Implement Laporta's algorithm to solve the system up to $r = 4$ and $s = 1$ for sector 7 and its subsectors.

SOLUTION:

a) We choose to complete the family as follow

$$I(a_1, a_2, a_3, a_4, a_5) = \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\left[k_1^2\right]^{a_1} \left[k_2^2\right]^{a_2} \left[(k_1 - k_2 - p)^2 - m^2\right]^{a_3} \left[k_1 \cdot p\right]^{a_4} \left[k_2 \cdot p\right]^{a_5}} .$$

Note that this is not a unique solution

b) The sectors 1-6 are zero

$$
\begin{aligned}
I(a_1,0,0,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2\big]^{a_1}}\,, \\
I(0,a_2,0,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_2^2\big]^{a_2}}\,, \\
I(a_1,a_2,0,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2\big]^{a_1}\big[k_2^2\big]^{a_2}}\,, \\
I(0,0,a_3,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[(k_1-k_2-p)^2-m^2\big]^{a_3}}\,, \\
I(a_1,0,a_3,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2\big]^{a_1}\big[(k_1-k_2-p)^2-m^2\big]^{a_3}}\,.
\end{aligned}
$$

c) From

$$
0 = \int [\mathrm{d}k_1][\mathrm{d}k_2] \partial_{k_1^\mu} \left(k_1^\mu \frac{1}{\big[k_1^2\big]^{a_1}\big[k_2^2\big]^{a_2}\big[(k_1-k_2-p)^2-m^2\big]^{a_3}\big[k_1\cdot p\big]^{a_4}\big[k_2\cdot p\big]^{a_5}} \right)
$$

we find

$$
0 = (d-2a_1-a_3-a_4) - a_3\mathbf{1}^-\mathbf{3}^+ + a_3\mathbf{2}^-\mathbf{3}^+ + 2a_3\mathbf{5}^-\mathbf{3}^+\,.
$$

with $a_1=a_2=a_3=1$ and $a_4=a_5=0$, we find our relation

$$
\begin{aligned}
0 &= (d-3)I(1,1,1,0,0) - I(0,1,2,0,0) + I(1,0,2,0,0) + 2I(1,1,2,0,-1) \\
\Rightarrow I(1,1,2,0,-1) &= \frac{3-d}{2} I(1,1,1,0,0)\,.
\end{aligned}
$$

d) We find

$$
\begin{aligned}
0 &= -2a_1 - a_3 - a_4 + d - a_3\mathbf{1}^-\mathbf{3}^+ + a_3\mathbf{2}^-\mathbf{3}^+ + 2a_3\mathbf{5}^-\mathbf{3}^+ \\
0 &= -a_1 + a_3 - a_1\mathbf{2}^-\mathbf{1}^+ + a_1\mathbf{3}^-\mathbf{1}^+ + 2a_1\mathbf{4}^-\mathbf{1}^+ - 2a_1\mathbf{5}^-\mathbf{1}^+ - a_3\mathbf{1}^-\mathbf{3}^+ \\
&\qquad + a_3\mathbf{2}^-\mathbf{3}^+ + 2a_3\mathbf{4}^-\mathbf{3}^+ - a_4\mathbf{5}^-\mathbf{4}^+ \\
0 &= -2a_1\mathbf{4}^-\mathbf{1}^+ - 2a_3\mathbf{4}^-\mathbf{3}^+ + 2a_3\mathbf{5}^-\mathbf{3}^+ + 2a_3m^2\mathbf{3}^+ - a_4m^2\mathbf{4}^+ \\
0 &= -a_2 + a_3 - a_2\mathbf{1}^-\mathbf{2}^+ + a_2\mathbf{3}^-\mathbf{2}^+ + 2a_2\mathbf{4}^-\mathbf{2}^+ - 2a_2\mathbf{5}^-\mathbf{2}^+ + a_3\mathbf{1}^-\mathbf{3}^+ \\
&\qquad - a_3\mathbf{2}^-\mathbf{3}^+ - 2a_3\mathbf{5}^-\mathbf{3}^+ - a_5\mathbf{4}^-\mathbf{5}^+ \\
0 &= -2a_2 - a_3 - a_5 + d + a_3\mathbf{1}^-\mathbf{3}^+ - a_3\mathbf{2}^-\mathbf{3}^+ - 2a_3\mathbf{4}^-\mathbf{3}^+ \\
0 &= -2a_2\mathbf{5}^-\mathbf{2}^+ + 2a_3\mathbf{4}^-\mathbf{3}^+ - 2a_3\mathbf{5}^-\mathbf{3}^+ - 2a_3m^2\mathbf{3}^+ - a_5m^2\mathbf{5}^+
\end{aligned}
$$

e) The implementation can be found online:
`https://gitlab.com/yannickulrich/loop-integration/-/blob/root/code/sheet2.m`

Exercise 2: Sunset diagram using computer codes

Consider again the same integral (1) but now with $p^2 = s \neq m^2$. Perform the reduction using `reduze` or `kira` for all integrals with $r \leq 4$ and $s \leq 1$.

a) Make a list of all 75 integrals you want to calculate using a computer program.

b) Perform the reduction of all integrals without specifying a basis of master integrals. How many master integrals do you find?

c) Consider the following possible choices of master integrals. Which ones do you prefer and why?

$$\vec{I}_1 = \begin{pmatrix} I(1,1,1;1) \\ I(1,1,2;1) \end{pmatrix}, \qquad \vec{I}_2 = \begin{pmatrix} I(1,1,1;1) \\ I(1,1,1;p\cdot k_1) \end{pmatrix}, \qquad \vec{I}_3 = \begin{pmatrix} I(1,2,1;1) \\ I(1,1,2;1) \end{pmatrix}. \tag{3}$$

SOLUTION:

a) We can use the following Mathematica program

```
RST[sunset[a__]] := {
  (* r *) Total[Select[{a}, # > 0 &]],
  (* s *) -Total[Select[{a}, # < 0 &]],
  (* t *) Length[Select[{a}, # > 0 &]]
}
IsInRange[{r_, s_, t_}] := 0 <= r <= 4 && s <= 1 && t >= 2
request = Select[
  Flatten@Outer[
    sunset,
    {-1, 0, 1, 2}, (* propagator 1 *)
    {-1, 0, 1, 2}, (* propagator 2 *)
    {-1, 0, 1, 2}, (* propagator 3 *)
    {0, -1}, (* auxilary propagator 1 *)
    {0, -1} (* auxilary propagator 1 *)
  ],
  IsInRange@*RS
];
Export["req.txt", request]
```

b) To set up `reduze` and `kira` we need to first specify the kinematics as `config/kinematics.yaml`

```
kinematics:
  incoming_momenta: [p]
  outgoing_momenta: [q]
  momentum_conservation: [q,p]
  kinematic_invariants:
     - [s, 2]
     - [m2, 2]
```

```
  scalarproduct_rules:
    - [[p,p], "s"]
```

and the integral family as `config/kinematics.yaml`

```
integralfamilies:
  - name: "sunset"
    loop_momenta: [k1, k2]
    propagators:
      - [ "k1", "0"]
      - [ "k2", "0"]
      - [ "k1-k2-p",  "m2"]
      - {  bilinear: [["k1","p"], "0"] }
      - {  bilinear: [["k2","p"], "0"] }
```

Next we specify the job we want to execute

```
jobs:
  - setup_sector_mappings: {}
  - reduce_sectors:
      sector_selection:
        select_recursively:
          - [sunset, 7]
      identities:
        ibp:
          - { r: [t, 5], s: [0, 2] }
  - select_reductions:
      input_file: req.txt
      output_file: tmp/req.txt.tmp
  - reduce_files:
      equation_files: ["tmp/req.txt.tmp"]
      output_file: "tmp/req2.txt.sol"
      # preferred_masters_file: "masters2.m"
  - export:
      input_file: tmp/req2.txt.sol
      output_file: req2.sol.m
      output_format: mma
      # preferred_masters_file: "masters2.m"
```

c) To change the basis we specify `preferred_masters_file` and rerun the reduction. Comparing the results we find that

- $\vec{I}_1$ introduces denominators of the form $d-4$ meaning that we need to know the masters higher than $\mathcal{O}(\epsilon^0)$
- $\vec{I}_2$ complicates the calculation of the masters
- $\vec{I}_3$ introduces complicated denominators of the form $(3-d)(8-3d)$ which is not ideal.