

Practical guide to loop integration

Exercise Sheet 1

Exercise 1: Γ functions

Use our basic integral

$$\int_0^\infty dx x^\alpha (a + bx)^\beta = \frac{\Gamma(1 + \alpha)\Gamma(-1 - \alpha - \beta)}{\Gamma(-\beta)} \frac{a^{1+\alpha+\beta}}{b^{1+\alpha}} \quad (1)$$

to calculate

a) $I_1(\alpha, \beta, \gamma, \delta) = \int dx_1 dx_2 dx_3 \delta(\dots) x_1^\alpha x_2^\beta x_3^\gamma (x_2 x_3 + x_1 x_2 + x_1 x_3)^\delta$

b) $I_2(\alpha, \beta, \gamma) = \int dx_1 dx_2 dx_3 dx_4 \delta(\dots) x_1^\alpha (x_2 x_3 + x_2 x_4 + x_3 x_4)^\beta (x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + x_3 x_4)^\gamma$

c) $I_2(-2\epsilon, -1 - 2\epsilon, -1 + 3\epsilon)$ up to $\mathcal{O}(\epsilon^0)$

SOLUTION:

- a) The order of integration does not matter in this example. Let us pick x_1 , x_2 , and finally x_3 with the δ function

$$\begin{aligned} I_1(\alpha, \beta, \gamma, \delta) &= \int dx_1 dx_2 dx_3 \delta(\dots) x_1^\alpha x_2^\beta x_3^\gamma (x_2 x_3 + x_1 x_2 + x_1 x_3)^\delta \\ &= \frac{\Gamma(\alpha + 1)\Gamma(-\alpha - \delta - 1)}{\Gamma(-\delta)} \int dx_2 dx_3 \delta(\dots) (x_2 + x_3)^{-\alpha-1} x_2^{\alpha+\beta+\delta+1} x_3^{\alpha+\delta+\gamma+1} \\ &= \frac{\Gamma(-\alpha - \delta - 1)\Gamma(-\beta - \delta - 1)\Gamma(\alpha + \beta + \delta + 2)}{\Gamma(-\delta)} \underbrace{\int dx_3 \delta(1 - x_3) x_3^{\alpha+\beta+2\delta+\gamma+2}}_1 \end{aligned}$$

- b) This example is slightly more complicated. Right now we cannot integrate eg. x_2 . By first

solving x_1 , we can then solve x_2 , and finally use the δ function for either x_3 or x_4

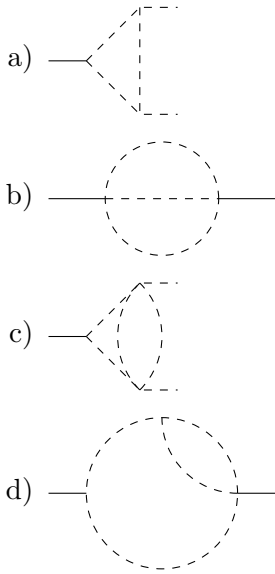
$$\begin{aligned}
I_2(\alpha, \beta, \gamma) &= \int dx_1 dx_2 dx_3 dx_4 \delta(\dots) x_1^\alpha \\
&\quad (x_2 x_3 + x_2 x_4 + x_3 x_4)^\beta (x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + x_3 x_4)^\gamma \\
&= \frac{\Gamma(1 + \alpha) \Gamma(-1 - \alpha - \gamma)}{\Gamma(-\gamma)} \\
&\quad \int dx_2 dx_3 dx_4 \delta(\dots) (x_3 + x_4)^{-1 - \alpha} (x_2 x_3 + x_2 x_4 + x_3 x_4)^{\beta + 1 + \alpha + \gamma} \\
&= \frac{\Gamma(\alpha + 1) \Gamma(-\alpha - \gamma - 1) \Gamma(-\alpha - \beta - \gamma - 2)}{\Gamma(-\gamma) \Gamma(-\alpha - \beta - \gamma - 1)} \\
&\quad \int dx_3 dx_4 \delta(\dots) x_3^{\alpha + \beta + \gamma + 2} x_4^{\alpha + \beta + \gamma + 2} (x_3 + x_4)^{-\alpha - 2} \\
&= \frac{\Gamma(\alpha + 1) \Gamma(-\alpha - \gamma - 1) \Gamma(-\beta - \gamma - 1) \Gamma(-\alpha - \beta - \gamma - 2) \Gamma(\alpha + \beta + \gamma + 3)}{\Gamma(\alpha + 2) \Gamma(-\gamma) \Gamma(-\alpha - \beta - \gamma - 1)}
\end{aligned}$$

c)

$$I_2(-2\epsilon, -1 - 2\epsilon, -1 + 3\epsilon) = -\frac{1}{\epsilon^2} - \frac{2}{\epsilon} - 8 + \frac{\pi^2}{2}$$

Exercise 2: Graph polynomials

Find the graph polynomials and calculate the loop integrals



Dashed lines represent massless propagators and solid lines massive ones.

SOLUTION:

a) We use the graphical method. At $\ell = 1$ this means one cut for \mathcal{U} and two for \mathcal{F}

$$\begin{aligned}
 \mathcal{U} &= \underbrace{\text{triangle}}_{x_1} + \underbrace{\text{triangle}}_{x_2} + \underbrace{\text{triangle}}_{x_3}, \\
 \mathcal{F} &= \underbrace{\text{figure-eight}}_{p^2 x_1 x_2} + \underbrace{\text{figure-eight}}_{q^2 x_1 x_3} + \underbrace{\text{figure-eight}}_{-s x_2 x_3} = -s x_2 x_3.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 I_1 &= -\Gamma(1 + \epsilon)\Gamma(1 - \epsilon) \int dx_1 dx_2 dx_3 \delta(\dots) \frac{(x_1 + x_2 + x_3)^{-1+2\epsilon}}{(-s)^{1+\epsilon} x_2^{1+\epsilon} x_3^{1+\epsilon}} \\
 &= -(-s)^{-1-\epsilon} \frac{\Gamma(1 - \epsilon)\Gamma(1 + \epsilon)\Gamma(-\epsilon)^2}{\Gamma(1 - 2\epsilon)}.
 \end{aligned}$$

b) At $\ell = 2$, we need two cuts for \mathcal{U} and three for \mathcal{F}

$$\mathcal{U} = \underbrace{\text{---} \bigcirc \text{---}}_{x_1 x_2} + \underbrace{\text{---} \bigcirc \text{---}}_{x_2 x_3} + \underbrace{\text{---} \bigcirc \text{---}}_{x_1 x_3},$$

$$\mathcal{F} = \underbrace{\text{---} \bigcirc \text{---}}_{-s x_1 x_2 x_3}.$$

Hence, the integral is

$$I_2 = -\Gamma(1-\epsilon)^2 \Gamma(2\epsilon-1) \int dx_1 dx_2 dx_3 \delta(\dots) \frac{(x_1 x_2 + x_2 x_3 + x_1 x_3)^{-3+3\epsilon}}{(-s)^{1+2\epsilon} x_2^{1+2\epsilon} x_3^{1+2\epsilon}}$$

$$= -(-s)^{1-2\epsilon} \frac{\Gamma(1-\epsilon)^5 \Gamma(-1+2\epsilon)}{\Gamma(3-3\epsilon)}.$$

c) Using the same methods as above,

$$\mathcal{U} = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4,$$

$$\mathcal{F} = (-s)x_1 x_3 x_4 + (-s)x_2 x_3 x_4.$$

We set $\delta(1-x_1)$ and integrate first x_3 , then x_4 , and finally x_2

$$I_3 = \Gamma(1-\epsilon)^2 \Gamma(2\epsilon) \int dx_1 dx_2 dx_3 dx_4 \delta(\dots) \frac{(x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4)^{-2+3\epsilon}}{(-s)^{2\epsilon} (x_1 + x_2)^{2\epsilon} x_3^{2\epsilon} x_4^{2\epsilon}}$$

$$= (-s)^{-2\epsilon} \frac{\Gamma(1-2\epsilon)^2 \Gamma(1-\epsilon)^4 \Gamma(\epsilon) \Gamma(2\epsilon)}{\Gamma(2-3\epsilon) \Gamma(2-2\epsilon)}.$$

d)

$$\mathcal{U} = x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + x_3 x_4,$$

$$\mathcal{F} = (-s)x_1 x_2 x_3 + (-s)x_1 x_2 x_4 (-s)x_1 x_3 x_4.$$

We set $\delta(1-x_4)$ and integrate first x_1 , then x_2 , and finally x_3

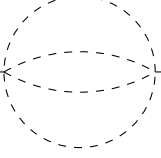
$$I_4 = \Gamma(1-\epsilon)^2 \Gamma(2\epsilon) \int dx_1 dx_2 dx_3 dx_4 \delta(\dots) \frac{(x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + x_3 x_4)^{-2+3\epsilon}}{(-s)^{2\epsilon} x_1^{2\epsilon} (x_2 x_3 + x_2 x_4 + x_3 x_4)^{2\epsilon}}$$

$$= (-s)^{-2\epsilon} \frac{\Gamma(1-2\epsilon) \Gamma(1-\epsilon)^5 \Gamma(\epsilon) \Gamma(2\epsilon)}{\Gamma(2-3\epsilon) \Gamma(2-2\epsilon) \Gamma(1+\epsilon)}.$$

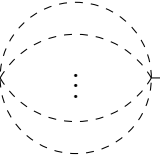
Exercise 3: Sunset integral

For general masses the sunset integral of 2b cannot be solved at two-loop. However, in the example above with massless internal lines and $p^2 \neq 0$ it was rather easy.

- a) This is also true at three loop. Show that

$$I_{\ell=3} = \text{---} \left(\text{---} \text{---} \text{---} \right) \text{---} = (-p^2)^{2-3\epsilon} \frac{\Gamma(1-\epsilon)^7 \Gamma(-2+3\epsilon)}{\Gamma(4-4\epsilon)}.$$


- b) Find a recurrence relation for the ℓ -loop graph polynomials \mathcal{F}_ℓ and \mathcal{U}_ℓ as a function of $\mathcal{F}_{\ell-1}$ and $\mathcal{U}_{\ell-1}$. Then construct the ℓ -loop sunset I_ℓ , integrate over one Feynman parameter, and relate the result to $I_{\ell-1}$. Finally, solve the resulting recurrence relation to obtain

$$I_\ell = \text{---} \left(\text{---} \text{---} \text{---} \text{---} \right) \text{---} = (-1)^{\ell+1} (-p^2)^{\ell-1-\ell\epsilon} \frac{\Gamma(1-\epsilon)^{2\ell+1} \Gamma(1-\ell(1-\epsilon))}{\Gamma((1+\ell)(1-\epsilon))}$$


- c) Write code to calculate I_ℓ using only the algorithm for \mathcal{U} and \mathcal{F} and the master formula. (*for the adventurous*)
- d) Use the optical theorem to relate $\Im I_\ell$ to the phase space volume for a $1 \rightarrow n$ decay

$$\int d\text{PS}^{1 \rightarrow n} = \frac{s^{n-2}}{2(4\pi)^{2n-3} \Gamma(n) \Gamma(n-1)}.$$

SOLUTION:

- a) We have

$$\begin{aligned} \mathcal{U}_{\ell=3} &= x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4, \\ \mathcal{F}_{\ell=3} &= (-p^2) x_1 x_2 x_3 x_4. \end{aligned}$$

For simplicity we set $p^2 = -1$. Hence,

$$I_{\ell=3} = \Gamma(1-\epsilon)^3 \Gamma(-2+3\epsilon) \int dx_1 dx_2 dx_3 dx_4 \delta(\dots) \mathcal{U}^{-4+4\epsilon} \mathcal{F}^{2-3\epsilon} = \frac{\Gamma(1-\epsilon)^7 \Gamma(-2+3\epsilon)}{\Gamma(4-4\epsilon)}.$$

- b) From what we have seen we can guess that

$$\begin{aligned} \mathcal{U}_\ell &= \prod_{i=1}^{\ell} x_i + \mathcal{U}_{\ell-1} x_{\ell+1}, \\ \mathcal{F}_\ell &= \prod_{i=1}^{\ell+1} x_i \end{aligned}$$

Hence, using our master formula with $r = \ell + 1$

$$\begin{aligned} I_\ell &= (-1)^{\ell+1} \Gamma(1-\epsilon)^\ell \Gamma(1-(1-\epsilon)\ell) \int dx_1 \cdots dx_{\ell+1} \delta(\cdots) \mathcal{U}^{(1+\ell)(\epsilon-1)} \mathcal{F}^{-1+(1-\epsilon)\ell} \\ &= (-1)^{\ell+1} \Gamma(1-\epsilon)^\ell \Gamma(1-(1-\epsilon)\ell) \int \left(\prod_{i=2}^{\ell+1} dx_i x_i^{-1+(1-\epsilon)\ell} \right) \left(\prod_{i=2}^{\ell} x_i + \mathcal{U}_{\ell-1} x_{\ell+1} \right)^{(1+\ell)(\epsilon-1)}, \end{aligned}$$

where we have used $\delta(1-x_1)$. Since $\mathcal{U}_{\ell-1}$ does not depend on $x_{\ell+1}$, we can solve this integration

$$\begin{aligned} I_\ell &= (-1)^{\ell+1} \Gamma(1-(1-\epsilon)\ell) \frac{\Gamma(1-\epsilon)^{\ell+1} \Gamma(\ell-\ell\epsilon)}{\Gamma((1-\epsilon)(1+\ell))} \int \left(\prod_{i=2}^{\ell} dx_i x_i^{(1+\ell)(\epsilon-1)} \right) \mathcal{U}_{\ell-1}^{\ell(\epsilon-1)} \\ &= -\Gamma(1-\epsilon)^2 \underbrace{\frac{\Gamma(1-\ell(1-\epsilon))}{\Gamma((1+\ell)(1-\epsilon))}}_{a_\ell} \underbrace{\frac{\Gamma(\ell-\ell\epsilon)}{\Gamma(1+(1-\ell)(1-\epsilon))}}_{1/a_{\ell-1}} I_{\ell-1}. \end{aligned}$$

This pattern of $a_\ell/a_{\ell-1}$ means that we can telescope the recurrence relation and find

$$\begin{aligned} I_\ell &= -\Gamma(1-\epsilon)^2 \frac{a_\ell}{a_{\ell-1}} I_{\ell-1} = -(-1)^\ell \Gamma(1-\epsilon)^{2\ell-2} \frac{a_\ell}{a_{\ell-1}} \frac{a_{\ell-1}}{a_{\ell-2}} \cdots \frac{a_2}{a_1} I_1 \\ &= -(-1)^\ell \Gamma(1-\epsilon)^{2\ell-2} \frac{a_\ell}{a_1} I_1. \end{aligned}$$

With $I_1 = \Gamma(1-\epsilon)^3 \Gamma(\epsilon) / \Gamma(2-2\epsilon)$ we arrive at

$$I_\ell = (-1)^{\ell+1} (-p^2)^{\ell-1-\ell\epsilon} \frac{\Gamma(1-\epsilon)^{2\ell+1} \Gamma(1-\ell(1-\epsilon))}{\Gamma((1+\ell)(1-\epsilon))}$$

after restoring the mass dimension of the integral.

c) We have the optical theorem

$$\int d\text{PS}^{1 \rightarrow (\ell+1)} = -2\Im I_\ell.$$

Restoring the $1/(16\pi^2)$ per loop order

$$\int d\text{PS}^{1 \rightarrow (\ell+1)} = -2 \left(\frac{1}{16\pi^2} \right)^\ell \Im \left((-s)^{\ell-1-\ell\epsilon} \frac{\Gamma(1-\epsilon)^{2\ell+1} \Gamma(1-\ell(1-\epsilon))}{\Gamma((1+\ell)(1-\epsilon))} \right)_{\epsilon \rightarrow 0}.$$

With $\Im(-s)^{a\epsilon} \sim a\pi\epsilon$ and $\Gamma(1-\ell(1-\epsilon)) \sim (-1)^{\ell+1}/\epsilon/\Gamma(1+\ell)$

$$\int d\text{PS}^{1 \rightarrow (\ell+1)} = 2 \left(\frac{1}{16\pi^2} \right)^\ell \ell \pi \epsilon (-s)^{\ell-1} \frac{(-1)^{\ell+1}}{\epsilon \Gamma(1+\ell)^2} = \frac{s^{\ell-1}}{2(4\pi)^{2\ell-1} \Gamma(1+\ell) \Gamma(\ell)}$$