Practical guide to loop integration
Exercise Sheet 6

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

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Exercise 1: Multidimensional Frobenius method In this problem we will explore the Frobenius method. Consider the following system of differential equations

$$\frac{\partial \vec{f}}{\partial x} = \underbrace{\begin{pmatrix} \frac{2-x}{1-x} & -\frac{1}{1-x} \\ -x & \frac{x}{1-x} \end{pmatrix}}_{M} \vec{f}\,. \tag{1}$$

A solution of this is given by

$$\vec{f} = \Big(\frac{\log(1-x)}{1-x}, \frac{1}{1-x} + \log(1-x)\Big)^T\,. \tag{2}$$

You may use this solution step-by-step to verify your working though in a real-life example, you would obviously not know it.

a) Construct the matrices $M^{(j)}$ for $j=0$, $j=1$, and $j=2$. Write down the matrices $\bar{M}$ and $\tilde{M}$

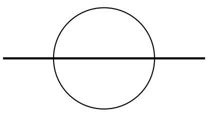
b) Find a vector $\vec{c}$ from the left-nullspace of $\tilde{M}$ such that $\vec{c}\tilde{M}=0$.

c) Make an ansatz for $f_1 = x^r \sum x^i a_i$ and calculate $\vec{c}\cdot(f_1, \partial_x f_1, \partial_x^2 f_1)$. Find the biggest r, a_1, a_2, and a_3.

d) Construct $\vec{f}$ by inverting $\tilde{M}$.

Exercise 2: Two-mass sunset diagram in $d = 2-2\epsilon$

Consider the following integral

$I_{\alpha\beta\gamma} =$ 

$$= \int \frac{[\mathrm{d}k_1][\mathrm{d}k_2]}{\big[k_1^2-m_1^2\big]^\alpha \big[k_2^2-m_1^2\big]^\beta \big[(k_1+k_2+p)^2-m_2^2\big]^\gamma}\,, \tag{3}$$

with $p^2 = m_2^2 \neq m_1^2$ in $d=2-2\epsilon$ dimensions. We use the integrals

$$\vec{I} = \big(m_2^2\big)^{4-d}\epsilon^2\Big(z^{-4}I_{220}, z^{-2}I_{202}, z^{-4}I_{211}, z^{-1}I_{112}\Big)^T \tag{4}$$

and the variable $z = m_2/m_1$.

a) Show that the differential equation matrix is precanonical

$$\partial_z \vec{I} = \begin{pmatrix} \frac{4\epsilon}{z} & 0 & 0 & 0 \\ 0 & \frac{2\epsilon}{z} & 0 & 0 \\ \frac{1}{z(1-z^2)} & -\frac{1}{z(1-z^2)} & \frac{z^2+(6-4z^2)\epsilon}{z(1-z^2)} & -\frac{1+2\epsilon}{z^2(1-z^2)} \\ -\frac{z^2}{1-z^2} & \frac{1}{1-z^2} & -\frac{z^2(1+2\epsilon)}{1-z^2} & \frac{z^2+2\epsilon}{z(1-z^2)} \end{pmatrix} \vec{I}\,. \tag{5}$$

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The boundary conditions can be fixed at $z = 1$

$$I_1 = I_2 = 1 + 2\epsilon + (1 + 2\zeta_2)\epsilon^2 + \mathcal{O}(\epsilon^3)\,, \tag{6a}$$

$$\begin{aligned} I_3 = I_4 = \frac{3}{8}\zeta_2 + \frac{1}{4} + \Big(+ \frac{21}{16}\zeta_3 - \frac{9}{4}\log 2\zeta_2 + \frac{9}{8}\zeta_2 - \frac{1}{2}\Big)\epsilon + \Big(- \frac{63}{16}\zeta_4 + 9\mathrm{Li}_4(\tfrac{1}{2}) + \frac{9}{2}\log 2^2\zeta_2 + \frac{3}{8}\log 2^4 \\ + \frac{63}{16}\zeta_3 - \frac{27}{4}\log 2\zeta_2 + \frac{1}{2}\zeta_2 + 1\Big)\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{6b}$$

We will now try to derive a series expression around $z = 1$.

b) Show (6). *(for the adventurous)*

c) Consider first the homogenous subsystem for $\vec{I}' = (I_3, I_4)$ with $x = 1 - z$

$$\partial_x \vec{I}' = \begin{pmatrix} \frac{1-x}{(x-2)x} & -\frac{1}{(x-2)(x-1)^2 x} \\ -\frac{(1-x)^2}{(x-2)x} & \frac{1-x}{(x-2)x} \end{pmatrix} \vec{I}'\,. \tag{7}$$

Construct $\tilde{M}$ and $\bar{M}$

$$\bar{M} = \begin{pmatrix} 1 & 0 \\ \frac{1-x}{(x-2)x} & -\frac{1}{(x-2)(x-1)^2 x} \\ \frac{2(x^2-2x+2)}{(x-2)^2 x^2} & \frac{2(3x^2-6x+2)}{(x-2)^2(x-1)^3 x^2} \end{pmatrix} \tag{8}$$

d) Find a solution $\vec{c}$ and write down the second-order differential equation for I_3. Make a Frobenius ansatz. What values of r are allowed?

e) Build a series solution for $r = 0$.

f) Construct the first-order differential equation to find the other solution for

$$\vec{c}_2 = \left(\frac{4}{x}\frac{1-x}{2-x}, 1 \right)\,. \tag{9}$$

g) Build a series solution for $r = -2$ and write down the full homogenous solution for the original differential equation.

h) Now we can consider the inhomogeneity. Show that for the k-th order in ϵ, it becomes

$$\vec{\mathcal{I}}^{(k)} = \frac{1}{x(x-2)}\left[\begin{pmatrix} \frac{1}{1-x} & -\frac{1}{1-x} \\ -(1-x)^2 & 1 \end{pmatrix} \cdot \left(I_1^{(k)}, I_2^{(k)}\right) + \begin{pmatrix} \frac{4x^2-8x-2}{x-1} & -\frac{2}{(x-1)^2} \\ -2(x-1)^2 & -\frac{2}{x-1} \end{pmatrix} \cdot \left(I_3^{(k-1)}, I_4^{(k-1)}\right) \right]. \tag{10}$$

i) Now construct a solution up to $\mathcal{O}(\epsilon^2)$. Evaluate the result at $z = 0.5$.