AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Practical guide to loop integration
Exercise Sheet 3

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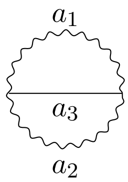
Exercise 1: Vacuum sunset

Calculate the vacuum sunset integral

$$I(a_1, a_2, a_3) =$$



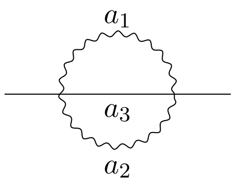
$$= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\left[k_1^2\right]^{a_1} \left[k_2^2\right]^{a_2} \left[(k_1 - k_2)^2 - m^2\right]^{a_3}} \tag{1}$$

for the cases $I(1, 2, 1)$ and $I(1, 1, 2)$ Why is this an easy integral to do?

Exercise 2: Off-shell sunset

Now consider the integral with an external leg

$$I(a_1, a_2, a_3) =$$



$$= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\left[k_1^2\right]^{a_1} \left[k_2^2\right]^{a_2} \left[(k_1 - k_2 - p)^2 - m^2\right]^{a_3}}\,, \tag{2}$$

with $p^2 = s = x\ m^2$.

a) Why can we not use the same calculation from the previous problem?

b) Show that this problem is solved by using the master integrals $\vec{I} = (I(1, 2, 1), I(1, 1, 2))^T$. These integrals are both starting at ϵ^{-2}.

c) Derive the differential equation matrices and show that

$$A_s = \begin{pmatrix} -\frac{(d-3)m^2+(d-4)s}{s(m^2-s)} & -\frac{(d-3)m^2}{s(m^2-s)} \\ \frac{d-4}{2s} & \frac{d-4}{2s} \end{pmatrix} \qquad \text{and} \qquad A_{m^2} = \begin{pmatrix} \frac{2d-7}{m^2-s} & \frac{d-3}{m^2-s} \\ -\frac{d-4}{2m^2} & \frac{d-4}{2m^2} \end{pmatrix}\,. \tag{3}$$

d) Find the mass scaling of each integral. Then rescale each integral such that it is finite and has vanishing mass scaling. Is the resulting system canonical?

e) Show that the matrix T achieves a canonical basis.

$$T = \frac{(m^2)^{-2\epsilon}}{\epsilon^2\, x}\left[\begin{pmatrix} -1 & 1-x \\ x & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 2 & -2(1-x) \\ -1-3x & 1-x \end{pmatrix}\right]. \tag{4}$$

f) Show that the boundary conditions $\vec{J}_0$ of the canonical system $\vec{J} = T^{-1}\vec{I}$ are

$$J_{0,1} = J_{0,2} = \frac{\epsilon\Gamma(1-\epsilon)^3\Gamma(\epsilon+1)\Gamma(2\epsilon-1)}{3\epsilon-1}. \tag{5}$$

g) Solve the remaining system up to ϵ^0 in $\vec{I}$.