Practical guide to loop integration
Exercise Sheet 1

AEC ALBERT EINSTEIN CENTER FOR FUNDAMENTAL PHYSICS

Summer Term 2024
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Issued: 16 April 2024

Exercise 1: Γ functions

Use our basic integral

$$\int_0^\infty \mathrm{d}x\; x^\alpha (a+bx)^\beta = \frac{\Gamma(1+\alpha)\Gamma(-1-\alpha-\beta)}{\Gamma(-\beta)}\frac{a^{1+\alpha+\beta}}{b^{1+\alpha}} \tag{1}$$

to calculate

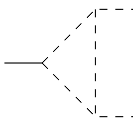
a) $I_1(\alpha,\beta,\gamma,\delta) = \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\delta(\cdots)x_1^\alpha x_2^\beta x_3^\gamma\big(x_2x_3+x_1x_2+x_1x_3\big)^\delta$

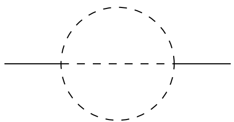
b) $I_2(\alpha,\beta,\gamma) = \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)x_1^\alpha\big(x_2x_3+x_2x_4+x_3x_4\big)^\beta\big(x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4\big)^\gamma$

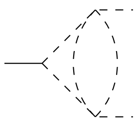
c) $I_2(-2\epsilon,-1-2\epsilon,-1+3\epsilon)$ up to $\mathcal{O}(\epsilon^0)$

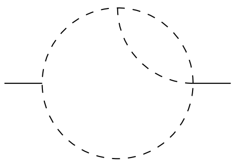
Exercise 2: Graph polynomials

Find the graph polynomials and calculate the loop integrals

a) 

b) 

c) 

d) 

Dashed lines represent massless propagators and solid lines massive ones.

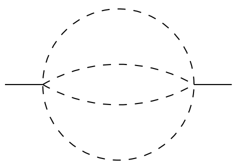
– please turn over –

Exercise 3: Sunset integral

For general masses the sunset integral of 2b cannot be solved at two-loop. However, in the example above with massless internal lines and $p^2 \neq 0$ it was rather easy.

a) This is also true at three loop. Show that

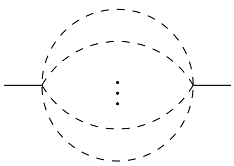
$$I_{\ell=3} =$$



$$= (-p^2)^{2-3\epsilon}\frac{\Gamma(1-\epsilon)^7\Gamma(-2+3\epsilon)}{\Gamma(4-4\epsilon)}\,.$$

b) Find a recurrence relation for the ℓ-loop graph polynomials $\mathcal{F}_\ell$ and $\mathcal{U}_\ell$ as a function of $\mathcal{F}_{\ell-1}$ and $\mathcal{U}_{\ell-1}$. Then construct the ℓ-loop sunset I_ℓ, integrate over one Feynman parameter, and relate the result to $I_{\ell-1}$. Finally, solve the resulting recurence relation to obtain

$$I_\ell =$$



$$= (-1)^{\ell+1}(-p^2)^{\ell-1-\ell\epsilon}\frac{\Gamma(1-\epsilon)^{2\ell+1}\Gamma\big(1-\ell(1-\epsilon)\big)}{\Gamma\big((1+\ell)(1-\epsilon)\big)}$$

c) Write code to calculate I_ℓ using only the algorithm for $\mathcal{U}$ and $\mathcal{F}$ and the master formula. *(for the adventurous)*

d) Use the optical theorem to relate $\Im I_\ell$ to the phase space volume for a $1 \to n$ decay

$$\int \mathrm{dPS}^{1\to n} = \frac{s^{n-2}}{2(4\pi)^{2n-3}\Gamma(n)\Gamma(n-1)}\,.$$