Practical guide to analytic loop integration

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Abstract

Loop integration is a vital part of any higher order calculation. However, practical tools to actually compute these integrals are rarely covered in lectures. In this course, I will cover some advanced tools that have been used in a number of multi-scale multi-loop calculations.

After introducing the problem, I will discuss integration-by-parts reduction to reduce the number of integrals. Next, I will discuss the method of regions to reduce the number of scales of the integrals. Finally, I will discuss the method of differential equations and in particular Auxilary Mass Flow to actually calculate the integrals.

The course will be composed of lectures introducing the techniques and practical, hands-on example at the two-loop level.

For further reading, see [1, 2, 3, 4, 5, 6].

Contents

1 Basics of loop integration

The calculation of Feynman integrals is a core problem in the calculation of higher order problems. Over the last decades, many techniques have been developed, both analytic and numeric. Many resources are available for commonly used techniques such as differential equations or sector decomposition. In this course, we will cover some more niche techniques that are useful for specialist application.

Definition 1 (Scalar loop integral)**.** We will almost exclusively consider integrals of form

$$\int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \cdots \frac{\mathrm{d}^d k_\ell}{(2\pi)^d} \frac{1}{\mathcal{P}_1^{\alpha_1} \cdots \mathcal{P}_t^{\alpha_t}}\,, \tag{1}$$

with ℓ loop momenta k_i. The powers α_i of the propagators $\mathcal{P}_i$ may be positive, negative, or zero. The propagators themselves are either linear or quadratic in the loop momenta. For example,

$$\mathcal{P}_i = \begin{cases} (k_i + p_i)^2 - m_i^2 + \mathrm{i}0^+ \\ 2k_i \cdot p_i - m_i^2 + \mathrm{i}0^+ \end{cases}\,, \tag{2}$$

where $\mathrm{i}0^+$ denotes the small imaginary part of the Feynman prescription required to ensure causality.

Definition 2 (Loop measure)**.** When solving loop integrals, we often encounter series expansions of the Γ function. This means that our intermediary results for the virtual matrix elements will contain terms proportional to the Euler constant $\gamma_E = -\Gamma(1)' = 0.5772...$. This is completely unphysical and will cancel once the virtual and real corrections are combined. It is, however, convenient to already drop these at the integral level. Hence, we re-define the loop measure as

$$[\mathrm{d}k] = \Gamma(1-\epsilon)\mu^{2\epsilon} \frac{\mathrm{d}^d k}{\mathrm{i}\pi^{d/2}}\,. \tag{3}$$

In what follows, I will often set $\mu = 1$ unless I want to make a specific point. Other definitions are used in the literature. If you are reusing results by other people make sure to check the conventions they have used.

Theorem 3 (Tadpole integral)**.** For $\ell = t = 1$ we have the tadpole integral

$$\int [\mathrm{d}k] \frac{1}{\big[k^2 - m^2 + \mathrm{i}0^+\big]^n} = (-1)^n \big(m^2\big)^{2-n-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(n-2+\epsilon)}{\Gamma(n)} \tag{4}$$

Proof. Our integrand has poles at $k_0 = \pm\sqrt{\vec{k}^2 + m^2} \mp \mathrm{i}0^+$ as shown in Figure 1. Since the poles are in the top-left and lower-right quadrant, the integral over the drawn contour vanishes. Since the integrand falls quickly enough, the integrals over the real and imaginary axis are equal up to a sign

$$0 = \oint \mathrm{d}k_0 = \left(\int_{-\infty}^{\infty} + \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} + \text{arcs} \right). \tag{5}$$

Substituting $k_0 \to \mathrm{i}k_0$ so that $k^2 = -k_0^2 - \vec{k}^2 = -k_E^2$ where k_E is a Euclidean momentum. This process in known as a Wick rotation, transforming our expression to

$$\begin{aligned} \int [\mathrm{d}k] \frac{1}{\big[k^2 - m^2 + \mathrm{i}0^+\big]^n} &= \frac{\Gamma(1-\epsilon)}{\pi^{d/2}} (-1)^n \int \mathrm{d}^d k_E \frac{1}{\big[k_E^2 + m^2 + \mathrm{i}0^+\big]^n} \\ &= \frac{\Gamma(1-\epsilon)}{\pi^{d/2}} (-1)^n \int \mathrm{d}\Omega_d \int_0^\infty \mathrm{d}k_E \frac{k_E^{d-1}}{\big[k_E^2 + m^2\big]^n}\,. \end{aligned} \tag{6}$$

At this point we can set $\mathrm{i}0^+ = 0$ since the denominator is always strictly bigger than zero. Using that the d-dimensional sphere has a volume of

$$\int \mathrm{d}\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \tag{7}$$

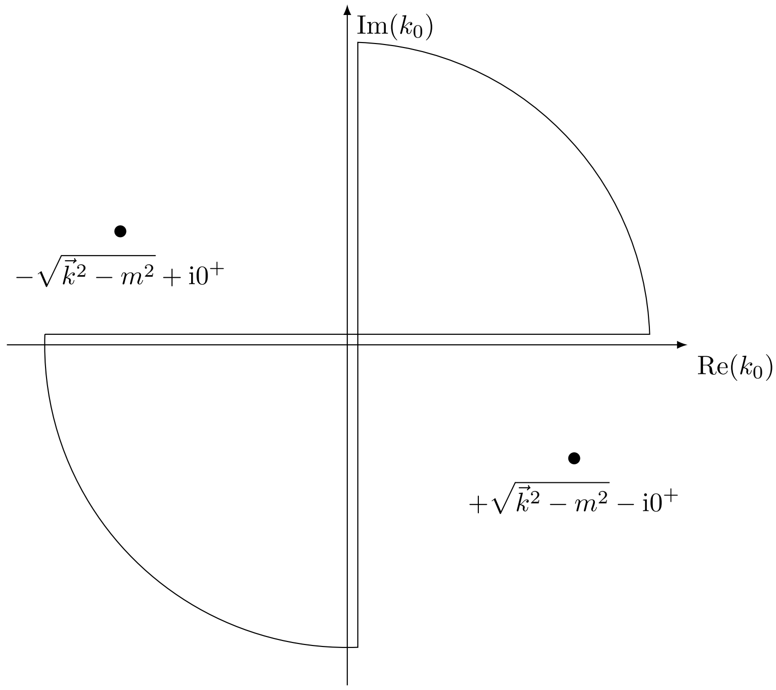


Figure 1: The poles of the k_0 intergation and the contour of (5)

and the basic integral (which is nothing but the definition of the B function)

$$\int_0^\infty \mathrm{d}x\; x^\alpha (a+bx)^\beta = \frac{\Gamma(1+\alpha)\Gamma(-1-\alpha-\beta)}{\Gamma(-\beta)}\frac{a^{1+\alpha+\beta}}{b^{1+\alpha}}\,, \tag{8}$$

we find

$$\begin{aligned}\int [\mathrm{d}k] \frac{1}{\big[k^2 - m^2 + \mathrm{i}0^+\big]^n} &= \frac{\Gamma(1-\epsilon)}{\pi^{d/2}}(-1)^n \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\big(m^2\big)^{d/2-n}\frac{\Gamma(\frac{d}{2})\Gamma(n-\frac{d}{2})}{2\Gamma(n)} \\ &= (-1)^n\big(m^2\big)^{2-n-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(n-2+\epsilon)}{\Gamma(n)}\,.\end{aligned} \tag{9}$$

□

To solve any loop integral with more propagators than loops, we could either solve the loops one by one or all in one go. Both methods are equivalent though difficult to relate in practical examples. We will be focusing on the latter case as the former can be viewed as a sub-class.

Lemma 4 (Feynman parametrisation & Cheng-Wu theorem [7])**.** We write

$$\frac{1}{\mathcal{P}_1^{\alpha_1}\cdots\mathcal{P}_t^{\alpha_t}} = \frac{\Gamma(r)}{\prod_j \Gamma(\alpha_j)}\int_0^\infty \prod_{j=1}^t \mathrm{d}x_j\; x_j^{\alpha_j-1}\delta\left(\sum_{i\in\nu} x_i - 1\right)\frac{1}{\Big(\mathcal{P}_1 x_1 + \ldots + \mathcal{P}_t x_t\Big)^r}\,. \tag{10}$$

where $r = \sum_j \alpha_j$ and ν a non-empty subset of $\{1,...,t\}$ (Cheng-Wu theorem).

Note that most books on QFT will assume $\nu = \{1,...,t\}$, reducing the integration region to $[0,1]\times[1,1-x_1]\times ....$ However, for analytic calculations we have found that having just one element, say $i=1$, in $\nu=\{1\}$ is a better choice, setting one $x_1 = 1$ and keeping the integration bounds at $[0,\infty]^{t-1}$.

Theorem 5. We can calculate a general ℓ-loop integral in terms of graph polynomials or Symanzik polynomials

$$I = (-1)^r \Gamma(1-\epsilon)^\ell \frac{\Gamma(r-\ell d/2)}{\prod_j \Gamma(\alpha_j)}\int_0^\infty \prod_{j=1}^t \mathrm{d}x_j\; x_j^{\alpha_j-1}\delta\Big(x_i - 1\Big)\underbrace{\frac{\mathcal{U}^{r-(\ell+1)d/2}}{(\mathcal{F}-\mathrm{i}0^+)^{r-\ell d/2}}}_{\mathcal{G}}\,. \tag{11}$$

$\mathcal{U}$ and $\mathcal{F}$ are the graph polynomials or Symanzik polynomials that can be computed analytically (see (17)), algorithmically (see Observation 6), or graph-theoretically (see Observation 7).

Further, $\mathcal{U} > 0$ so its prescription does not matter. $\mathcal{F}$, however, can be both positive and negative. It is hence important to properly include its prescription as $\mathcal{F} \to \mathcal{F} - \mathrm{i}0^+$.

Proof. We use (10) to write the dominator as

$$D = \mathcal{P}_1 x_1 + ... + \mathcal{P}_t x_t = k^T \cdot M(x_i) \cdot k - 2Q(x_i, q_j)^T k + J(x_i, s_{jk}) + \mathrm{i}0^+ \,, \tag{12}$$

with a $\ell \times \ell$ matrix M, ℓ-vectors $k = (k_1, ..., k_\ell)$ and Q, depending on the Feynman parameters x_i, external momenta q_j and invariants $s_{jk} = 2q_j \cdot q_k$. By shifting $k \to k + M^{-1}Q$ we cancel the linear term so that after diagonalising M (with eigenvalues λ_i) we have

$$D = k^T \cdot \mathrm{diag}(\lambda_i) \cdot k - \Delta \quad \text{with} \quad \Delta = Q^T M^{-1} Q - J - \mathrm{i}0^+ \,. \tag{13}$$

Δ has a $-\mathrm{i}0^+$ prescription since we started with $J + \mathrm{i}0^+$ in (12). Next, we rescale $k_i \to \lambda_i^{-1/2} k_i$ to factorise the loop integrations

$$I = \frac{\Gamma(r)}{\prod_j \Gamma(\alpha_j)} \int_0^\infty \prod_{j=1}^t \mathrm{d}x_j \; x_j^{\alpha_j - 1} \delta(\cdots) \underbrace{\int \lambda_1^{-d/2} [\mathrm{d}k_1] \cdots \lambda_\ell^{-d/2} [\mathrm{d}k_\ell] \frac{1}{\left[k^T \cdot k - \Delta\right]^r}}_{I'} \tag{14}$$

We can extend (4) to cover this case

$$\begin{aligned} I' &= \left(\frac{\Gamma(1-\epsilon)}{\pi^{d/2}}\right)^\ell (-1)^r \int \mathrm{d}^d k_{1,E} \cdots \mathrm{d}^d k_{\ell,E} \frac{1}{\left[k_{1,E}^2 + \cdots + k_{\ell,E}^2 + \Delta\right]^r} \\ &= (-1)^r \Delta^{\ell d/2 - r} \frac{\Gamma(1-\epsilon)^\ell \Gamma(r - \ell d/2)}{\Gamma(r)} \,. \end{aligned} \tag{15}$$

With this, we can write

$$I = \frac{\Gamma(r)}{\prod_j \Gamma(\alpha_j)} \int_0^\infty \prod_{j=1}^t \mathrm{d}x_j \; x_j^{\alpha_j - 1} \lambda_i^{-d/2} \delta(\cdots) (-1)^r \Delta^{-r + \ell d/2} \frac{\Gamma(1-\epsilon)^\ell \Gamma(r - \ell d/2)}{\Gamma(r)} \,. \tag{16}$$

Here we have used (4) to find the general Feynman-parametrised form of the ℓ-loop integral. We now identify

$$\mathcal{U} = \det M = \prod_j \lambda_j \,, \qquad \mathcal{F} = \det M \times \Delta \,, \tag{17}$$

to arrive at (11). □

Observation 6 (Algorithmic calculation of $\mathcal{U}$ and $\mathcal{F}$)**.** We can calculate $\mathcal{U}$ and $\mathcal{F}$ directly without having to manually find the eigenvalues λ_i. Instead, we calculate Δ and $\det M$ directly from (12). Collecting terms of k_1

$$(12) \equiv D_1 = k_1^2 \; t_1^{(2)} + k_1 \cdot t_1^{(1)} + t_1^{(0)} \,, \tag{18}$$

we identify $t_1^{(2)} \equiv \lambda_1$ and set

$$D_2 = t_1^{(0)} - \frac{t_1^{(1)} \cdot t_1^{(1)}}{4 t_1^{(2)}} \,. \tag{19}$$

We now repeat this and identify $t_2^{(i)}$

$$D_2 = k_2^2 \; t_2^{(2)} + k_2 \cdot t_2^{(1)} + t_2^{(0)} \tag{20}$$

in order to construct D_3. Eventually, we will reach $D_{\ell+1} = -\Delta$. This can be efficiently implemented in Mathematica as shown in Listing 2.

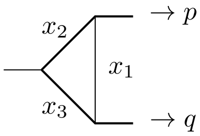
```
(* This function takes D_i and U_i == \lambda_1 * ... * \lambda_i and
   returns D_{i+1} and U_{i+1} *)
UFstep[{di_, ui_}, k_] := Module[{t0, t1, t2},
    t0 = Coefficient[di, k, 0];
    t1 = Coefficient[di, k, 1];
    t2 = Coefficient[di, k, 2];

    {t0 - t1*t1 / (4*t2), ui * t2}
]

(* This functions takes a list of loop momenta and propagators and
   folds UFstep over it *)
UF[ks_, props_] := Module[{d0, dl, ul},
    d0 = Total[props x /@ Range[Length[props]]];
    {dl, ul} = Fold[UFstep, {d0, 1}, ks];
    {ul, dl ul}
]
```

Listing 2: Mathematica implementation of $\mathcal{U}$ and $\mathcal{F}$

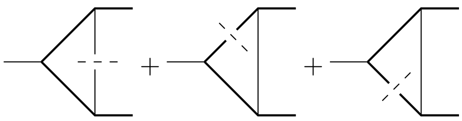
Observation 7 (Graph-theoretical determination of $\mathcal{U}$ and $\mathcal{F}$)**.** $\mathcal{U}$ and $\mathcal{F}$ can also be determined just by considering the graph of the loop integral in question. (12) assigns a Feynman parameter x_i to each edge of the graph. For example, consider the following one-loop integral



$$= \int [\mathrm{d}k_1] \frac{1}{k_1^2} \frac{1}{(k_1+p)^2 - m^2} \frac{1}{(k_1-q)^2 - m^2} \tag{21}$$

$\mathcal{U}$ is then determined by adding the products x_i for each ℓ line cuts of propagators that results in a tree-level

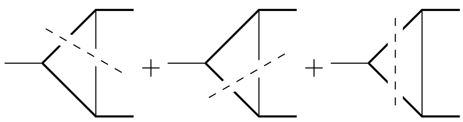
$\mathcal{U} =$



$$= x_1 + x_2 + x_3\,. \tag{22}$$

Similarly, for massless propagators, $\mathcal{F}$ is found by adding all $(\ell+1)$ line cuts with the momentum flowing through the resulting tree

$\mathcal{F}_0 =$



$$= p^2 x_1 x_2 + q^2 x_1 x_3 + (p+q)^2 x_2 x_3\,. \tag{23}$$

To account for massive propagators we write

$$\mathcal{F} = \mathcal{F}_0 + \sum x_i m_i^2 \mathcal{U}\,, \tag{24}$$

where m_i is the mass of the propagator assosciated to x_i.

Theorem 8 (Master formula for loop integrals)**.** An arbitrary integral, inlcuding numerators, can be calculated as

$$
\begin{aligned}
I = (-1)^r \Gamma(1-\epsilon)^\ell \Gamma(r-s-\ell d/2) \int_0^\infty \delta\Big(x_i - 1\Big) \left(\prod_{j=1}^{t} \mathrm{d}x_j \, \frac{x_j^{\alpha_j - 1}}{\Gamma(\alpha_j)} \right) \\
\left(\prod_{j=t+1}^{p} \frac{\partial^{-\alpha_j}}{\partial x_j^{-\alpha_j}} \right) \frac{\mathcal{U}^{r-s-(\ell+1)d/2}}{(\mathcal{F} - \mathrm{i}0^+)^{r-s-\ell d/2}} \Bigg|_{x_{t+1}=\ldots=x_p=0} ,
\end{aligned}
\tag{25}
$$

with r (s) the sum of positive (negative) indices, t the number of positive indices and p the length of the family as defined above. This implies that

$$
\alpha_1, ..., \alpha_t > 0 \quad \text{and} \quad \alpha_{t+1}, ..., \alpha_p \leq 0\,. \tag{26}
$$

Proof. As discussed above, we implement numerators in integrals by setting some $\alpha_i < 0$. However, that would make the Feynman parametrisation ill-defined because the Γ function diverges for negative integers. To solve this problem, we note an identity for Mellin transforms called Ramanujan's master theorem. In our language it states that the Mellin transform of a function $f(x)$ evaluated at negative integers $-n$ can be written as the n-th derivative of f

$$
\{\mathcal{M}f\}(-n) = \int \mathrm{d}x \, x^{-n-1} \; f(x) = \Gamma(-n) f^{(n)}(0)\,. \tag{27}
$$

Now the $\Gamma(-n)$ cancels, finally leading to our master formula. □

Lemma 9 (Lee-Pomeransky representation)**.** An alternative form of (11) is the Lee-Pomeransky representation

$$
I = \frac{\Gamma(d/2)}{\Gamma(\frac{l+1}{2}d - r) \prod_j \Gamma(\alpha_j)} \int \left(\prod_j \frac{\mathrm{d}x_j}{x_j} x_j^{\alpha_j} \right) \Big(\mathcal{U} + \mathcal{F} \Big)^{-d/2} . \tag{28}
$$

2 Integration-by-parts reduction

Real world calculations often involve many hundreds of integrals; computing these one by one is clearly infeasible. Instead we require a method to reduce the number of integrals to a manageable number of so-called master integrals that we can then calculate.

2.1 Organising integrals

Definition 10 (Integral family)**.** If we have ρ independent external momenta (after applying momentum conservation), we have

$$
\left(\!\!\binom{\rho}{2}\!\!\right) = \binom{\rho+1}{2} = \frac{(\rho+1)\rho}{2} \tag{29}
$$

possible ways to build scalar products (including masses p_i^2). Here, we have defined the multichoose function

$$
\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}\,, \tag{30}
$$

that counts the number of ways one can pick k unordered elements from a set of n elements, allowing for repetition.

If we have ℓ loop momenta, there are

$$
p = \left(\!\!\binom{\rho+\ell}{2}\!\!\right) - \left(\!\!\binom{\rho}{2}\!\!\right) = \ell \frac{1+\ell+2\rho}{2} \tag{31}
$$

possible scalar products of involving at least one loop momentum. A set of propagators of this size that allows all scalar products to be written through propagators is called a family. These propagators rarely all belong to the actual diagram and some might be fictitious, added purely to bring up the numbers.

```
(* We define t as the number of propagators,
             r as the sum of pos. powers,
             s as the sum of neg. powers *)
TRS[a_List] := {
    Length@Select[a, # > 0 &],
    Total[ Select[a, # > 0 &]],
    Total[-Select[a, # < 0 &]]
}
```

Listing 3: Mathematica implementation of t, r, and s given an integral

Definition 11 (Reducible scalar integral)**.** When calculating matrix elements, we often have to find integrals with numerators. Since the family is complete, we can always write these as propagators and bring the matrix element into the following form

$$\sum_n C_n \times \int \prod_{j=1}^{\ell} [\mathrm{d}k_j] \frac{1}{\mathcal{P}_{1,n}^{\alpha_{1,n}} \cdots \mathcal{P}_{p,n}^{\alpha_{p,n}}} \,. \tag{32}$$

The powers α_i of the propagators $\mathcal{P}_i$ may be negative or zero and the C_n are functions of the external kinematics and the dimension d. These integrals are referred to as reducible scalar integrals.

If we have scalar products with other momenta or even vector integrals we first have to use Passarino-Veltman decomposition as we would at one-loop.

Definition 12 (`reduze` organisation [8])**.** In virtually no case are all $\alpha_i > 0$. Hence, for a given scalar integral we define

- t: the number of $\alpha_i > 0$
- $r = \sum_{\alpha_i > 0} \alpha_i$ the sum of denominator powers
- $s = -\sum_{\alpha_i < 0} \alpha_i$ the sum of numerator powers
- the sector ID

$$\mathrm{ID} = \sum_{k=1}^{t} 2^{i_k - 1} \quad \text{with} \quad \alpha_{i_1}, ..., \alpha_{i_t} > 0 \,. \tag{33}$$

Obviously $s \geq 0$ and $r \geq t$. This serves to organise integrals because as soon as one integral in a sector can be calculated all integrals of the sector can be calculated, at least in principle.

Definition 13 (Corner integral)**.** For a given sector, we call an integral that has only unit powers the corner integral of that sector. It obviously has $r = t$.

2.2 Seed identities

To reduce the number of scalar integrals, we are using integration-by-parts (IBP) identities

$$\int \mathrm{d}x \; u \; v' = u \; v - \int \mathrm{d}x \; u' \; v \,. \tag{34}$$

Theorem 14 (IBP for loop integral)**.** Since the loop integration goes from $-\infty$ to ∞, one can show that the surface term $u \; v$ vanishes in dimensional regularisation. In the language of loop integrals

$$\int \prod_{j=1}^{\ell} [\mathrm{d}k_j] \frac{\partial}{\partial k_i} \cdot \left(q \, \frac{1}{\mathcal{P}_1^{\alpha_1} \cdots \mathcal{P}_t^{\alpha_t}} \right) = 0 \,, \quad i = 1, ..., \ell \,, \tag{35}$$

where q represents either a loop or an external momentum. Note that q is inside the derivative s.t. if $q = k_i$ the product rule has to be used on the integrand with

$$\frac{\partial}{\partial k} \cdot k \equiv \frac{\partial}{\partial k_\mu} k^\mu = d\,. \tag{36}$$

IBP relations now allow us to get identities between different integrals.

Example 15 (Heavy quark bubble)**.** Consider the following example

$$I(a,b) = \int [\mathrm{d}k] \frac{1}{\big[k^2\big]^a \big[(k-p)^2 - M^2\big]^b} \quad \text{with} \quad p^2 = M^2\,. \tag{37}$$

We can relate integrals with different a and b through

$$\begin{aligned} q = k: &\qquad 0 = (d - 2a - b) I(a,b) - b I(a-1, b+1)\,, \\ q = p: &\qquad 0 = (\;-\; a + b) I(a,b) - b I(a-1,b+1) + a I(a+1, b-1) + 2bM^2 I(a, b+1)\,. \end{aligned} \tag{38}$$

Definition 16 (Seed identity & compact notation)**.** We call identities of the form (38) seed identities because we will be using them to create more identities by choosing a and b.

We can write seed identities using the short-hand notation of [2]. $\mathbf{n}^\pm$ indicates that the power of the n-th propagator is raised (lowered) by one.

$$\begin{aligned} 0 &= d - 2a - b - b\, \mathbf{1}^- \mathbf{2}^+\,, \\ 0 &= \;-\; a + b - b\, \mathbf{1}^- \mathbf{2}^+ + a\, \mathbf{2}^- \mathbf{1}^+ + 2bM^2\, \mathbf{2}^+\,. \end{aligned} \tag{39}$$

Proof of Observation 15. Setting $q = k$, we apply (35)

$$\begin{aligned} i(a,b) &= \frac{\partial}{\partial k_\mu} \left(k^\mu \; \frac{1}{\big[k^2\big]^a \big[(k-p)^2 - M^2\big]^b} \right) \\ &= k^\mu \frac{\partial}{\partial k_\mu} \left(\frac{1}{\big[k^2\big]^a \big[(k-p)^2 - M^2\big]^b} \right) + \left(\frac{\partial}{\partial k_\mu} k^\mu \right) \frac{1}{\big[k^2\big]^a \big[(k-p)^2 - M^2\big]^b} \\ &= -2k^\mu \left(\frac{a\, k^\mu}{\big[k^2\big]^{1+a} \big[(k-p)^2 - M^2\big]^b} + \frac{b\,(k^\mu - p^\mu)}{\big[k^2\big]^a \big[(k-p)^2 - M^2\big]^{1+b}} \right) \\ &\qquad + \frac{d}{\big[k^2\big]^a \big[(k-p)^2 - M^2\big]^b} \\ &= \frac{(d - 2a - 2b)k^2 + 2(2a + b - d) k \cdot p}{\big[k^2\big]^a \big[(k-p)^2 - M^2\big]^{b+1}}\,. \end{aligned} \tag{40}$$

We now turn this expression back into scalar integrals of the form $I(a', b')$. After loop integration and setting $\int [\mathrm{d}k]\; i(a,b) = 0$ we finally have our first seed identity

$$0 = -bI(a-1, b+1) + (d - 2a - b) I(a,b)\,. \tag{41}$$

With $q = p$, we find the other identity.

□

2.3 Laporta algorithm

We now can use identities like the ones above to reduce integrals. However, doing this by hand is not practical. Instead, we use a modified version of Gaussian elimination to solve a very large though sparse system of linear equations by focussing on the difficult integrals first.

```
moms = {LTensor[k1, mu], LTensor[p, mu], LTensor[q, mu]};

ibps = Collect[ToFamily@Expand[
    D[moms FromFamily[int[{a1,a2,a3}]],k1]
]/.onshell,_int,Factor];
```

Listing 4: Mathematica implementation of the generation of seed identities

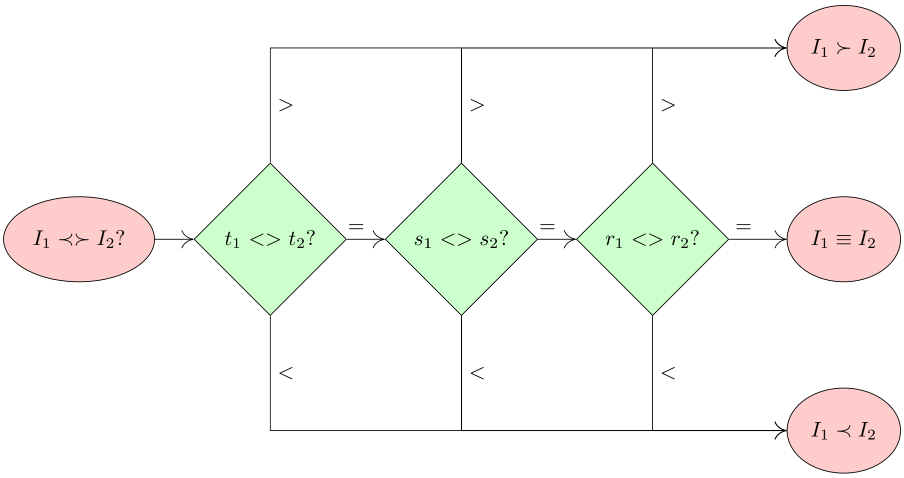


Figure 5: A common lexicographical ordering

Definition 17 (Lexicographic ordering)**.** Given two integrals, we need a way to decide which one is more complicated. The exact specification of this ordering does not matter as long as it is consistent. We will be using the ordering of Figure 2.3: given $I_1(t_1, r_1, s_1)$ and $I_2(t_2, r_2, s_2)$, it prefers small t, s, and r in that order.

Theorem 18 (Laporta's algorithm)**.** We can now reduce a set of irreducible integral to a (hopefully) small set of master integrals using Laporta's algorithm [9, 10] (Figure 6 and Listing 7)

1. Generate $i_{\max}$ seed identities of Definition 16
2. List all $j_{\max}$ reducible integrals up to some cut-off $r_{\max}$ and $s_{\max}$ and order them using the lexicographical ordering
3. Apply an identity to an integral and call the resulting relation $\mathcal{R}$
4. Substitute all known relations into $\mathcal{R}$, obtaining a relation $\mathcal{R}' = \sum_i c_i I_i = 0$
5. If $\mathcal{R}'$ is trivial, i.e. $c_i = 0$ and hence $0 = 0$, go back to Step 3. Otherwise, solve $\mathcal{R}'$ for the most complicated integral and remember this relation.
6. Go back to Step 3 until all integrals and seed identities have been used.
7. Re-substitute and clean up the relations

This algorithm converges because the cut-off points $r_{\max}$ and $s_{\max}$. The system is naturally overdetermined because the number of new integrals grows slower than the number of equations, resulting in many trivial relations along the way.

Observation 19. The implementation provided here is only for educational purposes as it is horribly inefficient. Instead, one should use one of many public codes such as `AIR` [11], `FIRE` [12], `Kira` [13], or `reduze` [8]. These codes essentially all implement Laporta's algorithm albeit much cleverer. For example, `Kira` uses finite-field sampling to improve the performance, especially for the back-substitution which can quickly become a bottleneck. Still, the IBP reduction is often a major bottleneck of higher order calculations and can run for months on a large cluster.

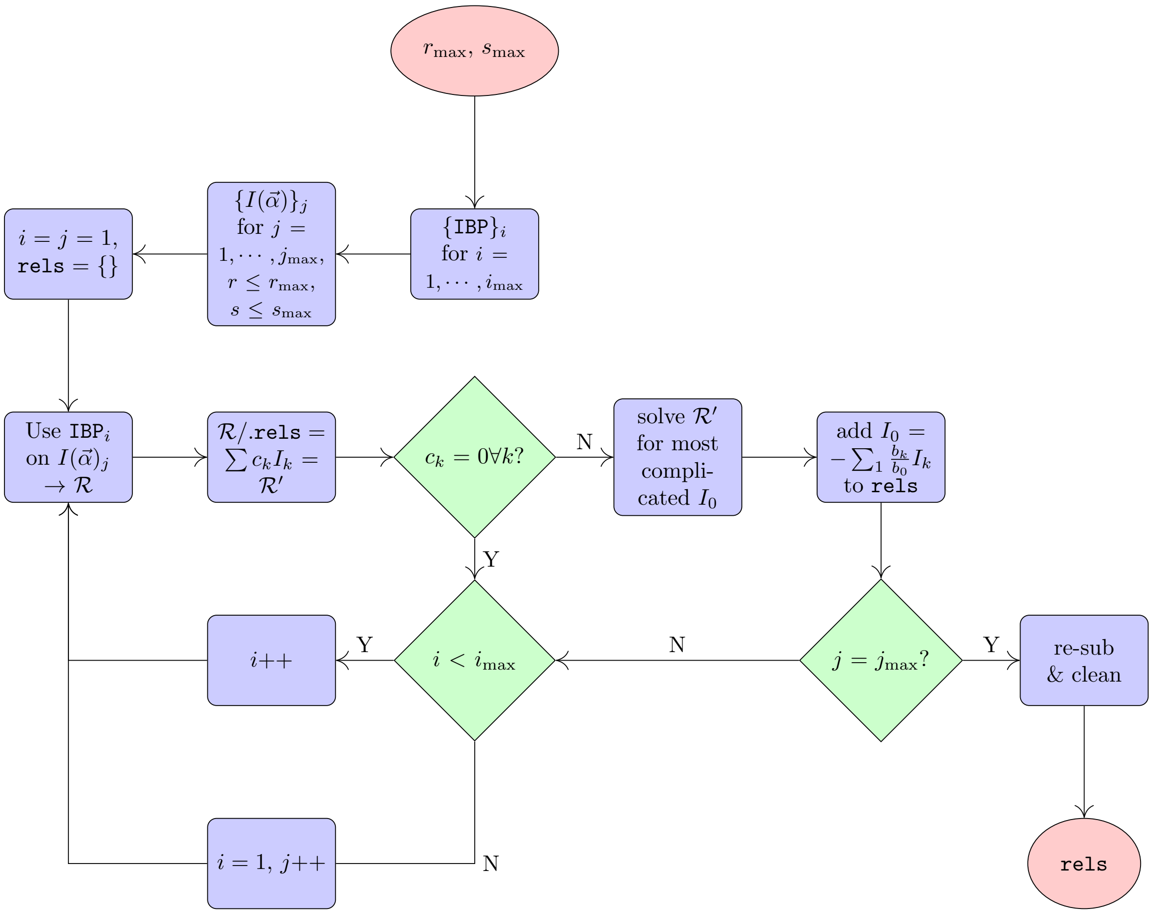


Figure 6: Flowchart for Laporta's algorithm

```
rmax = 5;
smax = 3;
IsOkay[{t_, r_, s_}] /; And[
    t > 0,
    rmax >= r,
    smax >= s
] = True;
IsOkay[__] = False;

intlist = Flatten[Outer[
    int[{##}]&,
    Sequence @@ ConstantArray[Range[-smax, rmax], Length[family]]
]];

seedPre = Sort[
    Select[intlist, IsOkay @* TRS],
    LexiOrdered
];
seedRep = seedPre /. int[a_] :> ReplaceAll[Thread[(ToExpression["a"<>
   ↪ ToString[#1]]  &/@ Range[Length[family]]) -> a]];

(* We now run Laporta's algorithm *)
rels = {i_int /; MemberQ[zerosectors, Sector[i]] -> 0};
Table[
    rel = Collect[seed[id] //. rels, _int, Factor];
    If[rel =!= 0,
        rels = Flatten[Join[rels, Solve[rel == 0, MostComplicated[rel
           ↪ ]]]]];
    ],
    {seed, seedRep[[ ;; ]]}, {id, ibps}
];

(* At the borders of the seed range we might get some useless
   identities that relate two complicated integrals we don't know. *)
IsUseless[a_ -> b_] := Quiet[
    LexiOrdered[MostComplicated[a], MostComplicated[b]] === 0
]
RemoveUseless[rels_] := Select[rels, Not@*IsUseless]

(* We also need to do the re-substitution and simplification *)
rels2 = RemoveUseless@Thread[
    rels[[2 ;;, 1]] -> Collect[
        rels[[2 ;;, 1]] //. rels, _int, Factor
    ]
];
```

Listing 7: Mathematica implementation of Laporta's algorithm

3 Method of differential equations

We have now a reduced our problem to a set of master integrals that we need to compute. While it is at least in principle possible to calculate one integral at a time using the representation in (25), this is rarely a good idea since we often have many hundreds of integrals and the required integrals cannot be solved analytically. Instead, we would prefer a method that can solve all master integrals in one stroke. The method of differential equations is the way to do this.

3.1 Deriving differential equations

Observation 20. Feynman integrals are functions of the dimensional regulator ϵ and masses and invariants collectively called s_i. Since derivatives w.r.t. the s_i can be related to derivates w.r.t. the momenta, it is easy to see the derivative of a Feynman integral is another Feynman integral in the same family. By IBP-reducing these back to the same set of master integrals, we can write a closed-form linear system of differential equations for the master integrals.

Theorem 21 (Derivatives of Feynman integrals)**.** The derivate of a set of Feynman integrals $\vec{I}$ w.r.t. some kinematic parameter s can be written as

$$\frac{\partial \vec{I}}{\partial s_i} = M_i(\{s\}, \epsilon)\vec{I} \tag{42}$$

where M_i is a matrix that depends on the kinematics and ϵ. The derivate ∂_{s_i} can be written in terms of the momenta

$$\frac{\partial}{\partial s_i} = \sum_{jk} a_{i,jk}\ p_k \cdot \frac{\partial}{\partial p_j}\,, \tag{43}$$

with coefficients $a_{i,jk}$ that are determined by applying this operator to the invariants themselves.

Example 22 (Heavy quark form factor)**.** Consider the family in 7

$$I_{\alpha\beta\gamma} = \int [\mathrm{d}k_1] \frac{1}{\big[k_1^2\big]^\alpha} \frac{1}{\big[(k_1+p)^2 - m^2\big]^\beta} \frac{1}{\big[(k_1-q)^2 - m^2\big]^\gamma} \tag{44}$$

with $p^2 = q^2 = m^2$ and $(p+q)^2 = s$. The derivative w.r.t. to s can be written as

$$\frac{\partial}{\partial s} = \Big(a_{s,11}\ p + a_{s,21}\ q\Big) \cdot \frac{\partial}{\partial p} + \Big(a_{s,12}\ p + a_{s,22}\ q\Big) \cdot \frac{\partial}{\partial q}\,. \tag{45}$$

By having this act on p^2, q^2, and $p \cdot q$, we find

$$\begin{aligned}
\frac{\partial(p^2)}{\partial s} &= \Big(a_{s,11}\ p + a_{s,21}\ q\Big) \cdot \underbrace{\frac{\partial(p^2)}{\partial p}}_{2p} + \Big(a_{s,12}\ p + a_{s,22}\ q\Big) \cdot \underbrace{\frac{\partial(p^2)}{\partial q}}_{0}\,,\\
\frac{\partial(q^2)}{\partial s} &= \Big(a_{s,11}\ p + a_{s,21}\ q\Big) \cdot \underbrace{\frac{\partial(q^2)}{\partial p}}_{0} + \Big(a_{s,12}\ p + a_{s,22}\ q\Big) \cdot \underbrace{\frac{\partial(q^2)}{\partial q}}_{2q}\,,\\
\frac{\partial(p\cdot q)}{\partial s} &= \Big(a_{s,11}\ p + a_{s,21}\ q\Big) \cdot \underbrace{\frac{\partial(p\cdot q)}{\partial p}}_{q} + \Big(a_{s,12}\ p + a_{s,22}\ q\Big) \cdot \underbrace{\frac{\partial(p\cdot q)}{\partial q}}_{p}\,.
\end{aligned} \tag{46}$$

This can be simplified to

$$\begin{aligned}
0 &= a_{s,11}\ 2m^2 + a_{s,21}\ \big(s - 2m^2\big)\,,\\
0 &= a_{s,22}\ 2m^2 + a_{s,12}\ \big(s - 2m^2\big)\,,\\
\frac{1}{2} &= \big(a_{s,11} + as,22\big)\ \frac{s-2m^2}{2} + \big(a_{s,21} + a_{s,12}\big)\ m^2\,.
\end{aligned} \tag{47}$$

When solving this under-determined linear system we have some choice. Here, we choose $a_{s,11} = 0$ and arrive at

$$\frac{\partial}{\partial s} = \frac{2m^2}{(4m^2-s)s}\, p\cdot\frac{\partial}{\partial q} + \frac{2m^2-s}{(4m^2-s)s}\, q\cdot\frac{\partial}{\partial q}\,. \tag{48}$$

A similar calculation with $a_{m^2,12} = a_{m^2,21}$, leads us to

$$\frac{\partial}{\partial m^2} = \frac{p^\mu - q^\mu}{4m^2-s}\Big(\frac{\partial}{\partial p^\mu} - \frac{\partial}{\partial q^\mu}\Big)\,. \tag{49}$$

We can now trivially calculate the effect of our two operators on an integral $I_{\alpha\beta\gamma}$. In the notation of Definition 16

$$\begin{aligned}\partial_s &= \frac{\gamma}{4m^2-s}\left(\frac{2m^2}{s}\mathbf{2}^-\mathbf{3}^+ - \mathbf{1}^-\mathbf{3}^+ + \frac{s-2m^2}{s}\right)\,,\\ \partial_{m^2} &= \frac{1}{4m^2-s}\Big(2\beta\mathbf{1}^-\mathbf{2}^+ - \beta\mathbf{3}^-\mathbf{2}^+ + 2\gamma\mathbf{1}^-\mathbf{3}^+ - \gamma\mathbf{2}^-\mathbf{3}^+ - \beta - \gamma\Big)\,.\end{aligned} \tag{50}$$

Example 23 (Master integrals in the heavy quark form factor)**.** We can now pick two integrals, eg. I_{001} and I_{011}, apply (50), perform the IBP reduction and arrive at our differential equations. For this it is useful to collect our two master integrals in a vector $\vec{I} = (I_{001}, I_{011})^T$

$$\begin{aligned}\frac{\partial\vec{I}}{\partial s} &= \frac{1}{(4m^2-s)s}\begin{pmatrix}0 & 0\\ 2-2\epsilon & -2m^2+s\epsilon\end{pmatrix}\vec{I}\,,\\ \frac{\partial\vec{I}}{\partial m^2} &= \frac{1}{(4m^2-s)m^2}\begin{pmatrix}(4m^2-s)(1-\epsilon) & 0\\ -2+2\epsilon & 2m^2(1-2\epsilon)\end{pmatrix}\vec{I}\,.\end{aligned} \tag{51}$$

Theorem 24 (Integral scaling)**.** When our integral depends on multiple scales s_i (such as s and m^2 in the above example), we have multiple matrices M_i. Note that

$$\sum_i s_i M_i = \text{diag}(\lambda_i)\,, \tag{52}$$

where the $\lambda_i = dl/2 - r_i + s_i$ are the scaling dimension of the i-th integral (cf. notation of Definition 12). This is an extremely useful cross check that the matrices M_i were correctly derived.

Theorem 25 (Some useful identities)**.** We often want to change variables and/or basis.

- We can change our basis of master integrals from $\vec{I}$ to $\vec{I}' = T^{-1}\vec{I}$ with some invertible matrix T. This means our M_i change to

$$M_i' = T^{-1}\cdot M_i\cdot T - T^{-1}\cdot\frac{\partial T}{\partial s_i} \tag{53}$$

- To change from the variable $s_i \to s_i'$, we write

$$M_{i'} = M_i \times \frac{\partial s_i}{\partial s_i'}\bigg|_{s_i\to s_i'} \tag{54}$$

- It is often advisable to choose scaleless variables and rescale the integrals so that all $\lambda_i = 1$.

3.2 Canonical basis and $d\log$ form

Example 26 (Change of variables)**.** Let us choose $T = \text{diag}(m^{-2\epsilon}, m^{-2-2\epsilon})/\epsilon$. This way our master integrals become

$$\vec{J} = \Big(m^{2\epsilon}\epsilon I_{001}, m^{2+2\epsilon}\epsilon I_{011}\Big)^T \tag{55}$$

and our matrices becomes

$$M'_s = \frac{1}{4m^2 - s}\begin{pmatrix} 0 & 0 \\ \frac{2m^2}{s}(1-\epsilon) & -\frac{2m^2}{s}+\epsilon \end{pmatrix},$$
$$M'_{m^2} = \frac{1}{4m^2 - s}\begin{pmatrix} \frac{4m^2-s}{m^2} & 0 \\ -2+2\epsilon & 6-\frac{s}{m^2}(1+\epsilon) \end{pmatrix}. \tag{56}$$

One can easily see that $sM'_s + m^2 M'_{m^2} = 1$. Let us now introduce $y = s/(4m^2)$. The new M_y is

$$M_y = \frac{1}{1-y}\frac{1}{2y}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + \frac{1}{1-y}\epsilon\begin{pmatrix} 0 & 0 \\ -\frac{1}{2y} & 1 \end{pmatrix}. \tag{57}$$

Definition 27 (Canonical form)**.** If all matrices M_i can be brought into the form $M_i = M_i^{(0)} + \epsilon M_i^{(1)}$ we call the basis *precanonical*. A *canonical basis* has further $M_i^{(0)} = 0$. In other words, the differential equation can be written as

$$\frac{\partial \vec{J}}{\partial s_i} = \epsilon M_i \vec{J}. \tag{58}$$

Once the differential equations are in canonical form, solving them is fairly straightforward, at least in principle.

Definition 28 ($d\log$ form)**.** In some cases it is further possible to simplify the equations in terms of differential forms

$$\mathrm{d}\vec{J} = \epsilon \mathrm{d}A\vec{J}, \tag{59}$$

where A can be written as

$$A = \sum_i A_i \log \eta_i \tag{60}$$

with $A_i \in \mathbb{Q}$ matrices of rational numbers. This is referred to as the $d\log$ form and it usually (but not always [14])[1] possible to write the answer in term of polylogarithms. The η_i are called letters and their set $\{\eta_i\}$ the alphabet. In practise this means that the original M can be written as

$$M = \sum_i \frac{A_i}{\eta_i}. \tag{61}$$

Example 29. There are algorithms that can assist in arriving at a canonical or precanonical form. However, the process often is a mixture of guesswork, experience, and computer codes. In our case, we can have

$$\vec{J} = \left(\epsilon\, m^{2\epsilon+2}\, I_{002}, \epsilon\sqrt{s}\sqrt{s-4m^2}\, m^{2\epsilon+2}\, I_{012}\right)^T, \tag{62}$$

with the matrix

$$M_s = \epsilon\begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{s}\sqrt{s-4m^2}} & -\frac{1}{s-4m^2} \end{pmatrix}. \tag{63}$$

The presence of squared propagators and square roots in $\vec{J}$ is a fairly common feature. To remove these, we perform one more change of variables $s/m^2 = -4x^2/(1-x^2)$ and arrive at

$$M_x = \epsilon\begin{pmatrix} 0 & 0 \\ \frac{1}{1+x}+\frac{1}{1-x} & \frac{1}{1+x}-\frac{1}{1-x} \end{pmatrix}. \tag{64}$$

Now our matrix is in $d\log$ form

$$\mathrm{d}\vec{J} = \epsilon\ \mathrm{d}\Big(A_1\log(1+x) + A_2\log(1-x)\Big)\vec{J} \tag{65}$$

$$A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \tag{66}$$

[1] This can eg. happen if the $d\log$ basis has multiple non-simultaneously rationalisable square roots.

3.3 Chen-interated integrals

Once our integral is in $d\log$ form, we are often done since our integral can now be solved in terms of Chen iterated integrals [15].

Definition 30 (Chen iterated integral)**.** The formal solution to (59) is a Chen iterated integral

$$\vec{J}(\vec{s},\epsilon) = \mathbb{P}\exp\left[\epsilon\int_\gamma \mathrm{d}A\right]\vec{J}_0(\epsilon)\,. \tag{67}$$

The $\mathbb{P}$ indicates path-ordering along the integration contour γ. The $\vec{J}_0(\epsilon)$ is the boundary condition of our differential equation.

Example 31 (Boundary bonditions of the heavy quark form factor at $x=0$)**.** Before we can solve the integral for arbitrary x, we need to solve it for a specific value of x to act as our boundary condition. For this we pick $x=0$ which corresponds to $s=0$. Since the original integral I_{012} is regular as $s\to 0$, our master integral is zero $I_{012}(s=0)=0$. The other integral I_{002} is a single-scale integral that could not be computed using differential equations anyway. However, it is trivial to see from (4) $I_{002} = -m^{-2\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)$. Hence, check sign

$$\vec{J}_0(\epsilon) = \Big(-m^2\Gamma(1-\epsilon)\Gamma(1+\epsilon), 0\Big)^T\,. \tag{68}$$

Definition 32 (Generalised polylogarithms)**.** The class of function that is obtained from iterated integrals of rational functions are called generalised or Goncharov polylogarithms (GPL) [16]

$$G(z_1,\cdots,z_m;y) = \int_0^y \frac{\mathrm{d}t_1}{t_1-z_1}\int_0^{t_1}\frac{\mathrm{d}t_2}{t_2-z_2}\cdots\int_0^{t_{m-1}}\frac{\mathrm{d}t_m}{t_m-z_m}\,. \tag{69}$$

These functions are extremely well studied and many tools exist to work with them [17, 18, 19, 20, 21]

Theorem 33. To turn the formal solution (67) into a practical solution, we expand it order-by-order

$$\vec{J}(\vec{s},\epsilon) = \sum_{n=0}^\infty \epsilon^n \vec{J}^{(n)}\,, \tag{70}$$

$$\begin{aligned}\vec{J}^{(0)}(\vec{s},\epsilon) &= \vec{J}_0^{(0)}\,,\\ \vec{J}^{(n)}(\vec{s},\epsilon) &= \sum_i\int_0^x \frac{\mathrm{d}x'}{\eta_i}\frac{\partial\log(\eta_i)}{\partial x'}A_i\cdot\vec{J}^{(n-1)} + \vec{J}_0^{(n)}\,.\end{aligned} \tag{71}$$

Example 34. We have with $m=1$

$$\vec{J}^{(0)} = \begin{pmatrix}-1\\0\end{pmatrix}\,, \tag{72}$$

$$\vec{J}^{(1)} = \int_0^x\frac{\mathrm{d}x'}{x'-1}\begin{pmatrix}0\\1\end{pmatrix} + \int_0^x\frac{\mathrm{d}x'}{x'+1}\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}0\\G(1;x)-G(-1;x)\end{pmatrix}\,, \tag{73}$$

$$\begin{aligned}\vec{J}^{(2)} &= \int_0^x\frac{\mathrm{d}x'}{x'-1}\begin{pmatrix}0\\G(1;x)-G(-1;x)\end{pmatrix} + \int_0^x\frac{\mathrm{d}x'}{x'+1}\begin{pmatrix}0\\G(1;x)-G(-1;x)\end{pmatrix} + \begin{pmatrix}-\zeta_2\\0\end{pmatrix}\\ &= \begin{pmatrix}-\zeta_2\\G(-1,1;x)-G(-1,-1;x)+G(1,1;x)-G(1,-1;x)\end{pmatrix}\,.\end{aligned} \tag{74}$$

We could continue with this expansion as long as we want to and hence arrive at an expression for our integral.

4 Method of regions

In general, the calculation of master integrals with full dependence of any parameter is very difficult and time consuming. However, in many cases this is not needed, often because the parameters have a strong hierarchy. This could either mean that the electron mass m^2 is much smaller than the momentum transfer Q^2 or that the W boson mass m_W^2 is much larger. In these cases, we instead calculate the integrals expanded in the small parameter (m/Q^2 or Q^2/m_W^2). The technique used to achieve this is the method of regions [22].

4.1 Momentum space

Example 35 (Heavy particle)**.** Let us first consider the case of a heavy particle $m_X^2 \gg s$ in

$$I = \int [\mathrm{d}k] \; \frac{1}{\big[k^2 - m_X^2 + \mathrm{i}0^+\big]\big[k^2 - 2k\cdot p + \mathrm{i}0^+\big]} \tag{75}$$

with $p^2 = m^2 \ll m_X^2$. The naive expansion of the integrand

$$\frac{1}{\big[k^2 - m_X^2 + \mathrm{i}0^+\big]\big[k^2 - 2k\cdot p + \mathrm{i}0^+\big]} = -\frac{1}{m_X^2}\frac{1}{\big[k^2 - 2k\cdot p + \mathrm{i}0^+\big]} + \mathcal{O}\Big(\frac{1}{m_X^4}\Big) \tag{76}$$

is obviously wrong since integration and expansion does not commute. The expansion is wrong in the region where $k \sim m_X$.

Theorem 36 (The method of regions in momentum space)**.** We can expand an integral already at integrand level by identifying all relevant regions, expanding in those up to whatever power we desire and adding them up. In principle infinitely many such regions exist, but most of them vanish in dimensional regularisation.

Proof. We only prove the example, though this argument can be formalised.

We split the integration at the scale Λ which is $m \ll \Lambda \ll m_X$

$$\begin{aligned} I &= \int_0^\Lambda [\mathrm{d}k] \; \frac{1}{\big[k^2 - m_X^2\big]\big[k^2 - 2k\cdot p\big]} + \int_\Lambda^\infty [\mathrm{d}k] \; \frac{1}{\big[k^2 - m_X^2\big]\big[k^2 - 2k\cdot p\big]} \\ &= \underbrace{-\frac{1}{m_X^2}\int_0^\Lambda [\mathrm{d}k] \; \frac{1}{\big[k^2 - 2k\cdot p\big]}}_{\text{soft}} + \underbrace{\int_\Lambda^\infty [\mathrm{d}k] \left(\frac{1}{\big[k^2 - m_X^2\big]\big[k^2\big]} + \frac{4(k\cdot p)^2}{\big[k^2 - m_X^2\big]^3\big[k^2\big]}\right)}_{\text{hard}} + \mathcal{O}\Big(\frac{1}{m_x^4}\Big) \end{aligned} \tag{77}$$

The first term is generally referred to the soft region and the second term as the hard region. We now need to relate these integrals back to what we can calculate. It turns out we can just integrate over the whole range since the difference is scaleless, eg.

$$\int_\Lambda^\infty [\mathrm{d}k] \; \frac{1}{\big[k^2 - 2k\cdot p\big]} \overset{m\ll\Lambda}{=} \sum_{i=0}^\infty \int_0^\infty [\mathrm{d}k] \; \frac{(m^2)^i}{\big[k^2\big]^{i+1}} = 0\,. \tag{78}$$

We can now solve the integrals in (77) over the whole range and obtain

$$I = \frac{1}{\epsilon} + 1 + \log\frac{\mu^2}{m_X^2} + \frac{m^2}{2m_X^2}\Big(1 + 2\log\frac{m^2}{m_X^2}\Big)\,. \tag{79}$$

This agrees with what we would obtain had we expanded the original integral after integration. □

Observation 37. This is directly connected to the concept of effective field theories (EFT) where we add new operators $\mathcal{Q}$ and Wilson coefficients $\mathcal{C}$ to our Lagrangian

$$\mathcal{L} \to \mathcal{L} + \sum_i \mathcal{C}_i \mathcal{Q}_i\,. \tag{80}$$

The hard region corresponds to the renormalisation of the Wilson coefficients while the soft region is the loop calculation in the EFT. In this language, the different orders in the expansion correspond directly to the dimensionality of the operators.

This type of expansion is very useful if we want to integrate out a heavy scale. However, we need more tools if we want to deal with light scales as well.

Definition 38 (Light-cone coordinates)**.** Let $p_i = E_i(1, \vec{n}_i\beta_i)$ be a high energetic particle ($\beta = 1-\mathcal{O}(\lambda^2)$), moving in the $\vec{n}_i$ direction. We now decompose its momentum into light-cone basis vectors $n_i = (1, \vec{n}_i)/\sqrt{2}$ and $\bar{n}_i = (1, -\vec{n}_i)/\sqrt{2}$.[2] We now write any momentum p_j as

$$p_j = (n_i\cdot p_j)\bar{n}_i + (\bar{n}_i\cdot p_j)n_i + p_j^{(\perp,i)} = p_j^{(+,i)} + p_j^{(-,i)} + p_j^{(\perp,i)} = (n_i\cdot p_j, \bar{n}_i\cdot p_j, p_{j,\perp})_i\,. \tag{81}$$

[2]Our definition of n and $\bar{n}$ differs from the standard convention by the normalisation factor $1/\sqrt{2}$.

For the case $i = j$ our original energetic particle p_i is now

$$p_i = \big(E_i(1-\beta_i)/\sqrt{2}, E_i(1+\beta_i)/\sqrt{2}, p_{i,\perp}\big)_i \sim (\lambda^2, 1, \lambda)_i\,. \tag{82}$$

We have especially $p_i^2 = m^2 \sim \lambda^2$ as planned

$$p_i^2 = 2p_i^{(+,i)} \cdot p_i^{(-,i)} + \big(p_i^{(\perp,i)}\big)^2 = m^2 \sim \lambda^2\,. \tag{83}$$

Here we have used that, since $\bar{n}^2 = n^2 = n \cdot p_\perp = 0$,

$$p_j^{(\pm,i)} \cdot p_k^{(\pm,i)} = p_j^{(\pm,i)} \cdot p_k^{(\perp,i)} = 0 \tag{84}$$

Definition 39 (Momentum regions)**.** Using light-cone coordinates, a region of the loop momentum k is then defined as a specific choice of parameters a, b, and c where $k \sim (\lambda^a, \lambda^b, \lambda^c)_i$ for a given direction n_i. At this point we expect an infinite number of regions corresponding to the infinite possible choices of a, b, and c.[3] Fortunately, almost all of the infinite number of regions turn out to be zero.

Common momentum regions that contribute are

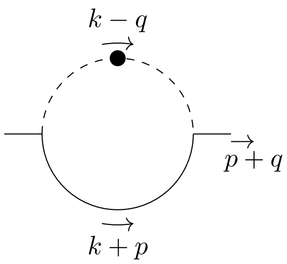
$$\text{hard:}\quad k \sim (1,1,1) \tag{85a}$$
$$\text{soft:}\quad k \sim (\lambda,\lambda,\lambda) \tag{85b}$$
$$\text{collinear:}\quad k \sim (\lambda^2,1,\lambda) \tag{85c}$$
$$\text{ultrasoft:}\quad k \sim (\lambda^2,\lambda^2,\lambda^2)\,, \tag{85d}$$

Note that these scalings are only defined for a given direction. If we have multiple directions of energetic particles as is often the case, we have to distinguish them, leading to more regions.

We might be able to reduce the number of directions if some particles are back-to-back as is the case in $\gamma^* \to e^+(n_1)e^-(n_2)$. The $(\lambda^2,1,\lambda)_2$ region is actually the same as the $(1,\lambda^2,\lambda)_1$ region, reducing the amount of bookkeeping required.

In what follows, I will drop the index for the direction when it is unambiguous.

Example 40 (Light particles)**.** Consider the following integral

$I =$ 

$$= \int [\mathrm{d}k] \frac{1}{\big[(k+p)^2 - M^2\big]\big[(k-q)^2 - m^2\big]^2}\,, \tag{86}$$

with $q^2 = m^2 \ll p^2 = M^2 \sim (p+q)^2 = s$. A convenient choice is $p \sim (1,1,0)$, $q \sim (0,1,\lambda)$ which results in

$$p^2 = 2p_+ \cdot p_- = M^2 \sim 1 \qquad \text{and} \qquad q^2 = q_\perp^2 = m^2 \sim \lambda^2\,. \tag{87}$$

By trying all sensible regions, we find that all but two are scaleless. Consider eg. $k \sim (\lambda,\lambda,\lambda)$

$$I_s = \frac{1}{\lambda^3} \int [\mathrm{d}k] \frac{1}{\big[2k_+ \cdot q_-\big]^2 \big[2k \cdot p\big]} = 0\,, \tag{88}$$

where we have used that $k_+ \cdot q_- = k \cdot q_-$. The only scaleful regions are $k \sim (1,1,1)$ (hard) and $k \sim (\lambda^2,1,\lambda)$ (collinear).

[3]One can show that $c = (a+b)/2$ with other choices resulting in scaleless integrals.

- In the hard region we have $k \sim (1,1,1)$ and we can write

$$I_h = \int [\mathrm{d}k] \frac{1}{\big[(k+p)^2 - M^2\big]\big[(k-q_-)^2\big]^2} + \mathcal{O}(\lambda^2)\,. \tag{89}$$

  The leading term is just the original integral with $m=0$ since $q_-^2 = 0$. In real-world applications, we might be able to obtain this from already known QCD results. We have

$$\begin{aligned} I_h &= \Gamma(1-\epsilon)\Gamma(\epsilon) \int \mathrm{d}x_1 \mathrm{d}x_2 \delta(\cdots) x_1 \frac{\big(x_1+x_2\big)^{-2+2\epsilon}}{\big(M^2 x_2^2 + (M^2-s)x_1 x_2\big)^\epsilon} \\ &= \Gamma(1-\epsilon)\Gamma(\epsilon) \int_0^1 \mathrm{d}x_2 (1-x_2) x_2^{-\epsilon} \big(M^2 - s + s x_2\big)^{-\epsilon} \\ &= \frac{(1-x)^\epsilon}{M^2}\Big(\frac{M^2}{\mu^2}\Big)^{-\epsilon}\Bigg(\frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{1-\epsilon}\,{}_2F_1\left[\begin{matrix}1-\epsilon,\ 1+\epsilon\\ 2-\epsilon\end{matrix};x\right] \\ &\qquad + \Gamma(1-\epsilon)\Gamma(\epsilon)\,{}_2F_1\left[\begin{matrix}-\epsilon,\ 1+\epsilon\\ 2-\epsilon\end{matrix};x\right]\Bigg) \\ &= \frac{1-x}{M^2}\Bigg(\frac{1}{\epsilon} + \Big(2+\frac{1}{x}\Big)\log(1-x) + \log\frac{\mu^2}{M^2}\Bigg) + \mathcal{O}(m^2)\,. \end{aligned} \tag{90}$$

  where we have set $\delta(1-x_1+x_2)$ and defined $x = s/(s-M^2)$. Here we have used a ${}_2F_1$ function that we will encounter again in Section 5 and expanded it with `HypExp` [23].

- In the collinear region, we have account for a factor λ^4 from d^4k. Hence, we have

$$\begin{aligned} I_c &= \int [\mathrm{d}k] \frac{1}{\big[2k_- \cdot p_+\big]\big[k^2 - 2k\cdot q\big]^2} \\ &= -\Gamma(1-\epsilon)\Gamma(1+\epsilon) \int \mathrm{d}x_1 \mathrm{d}x_2 \delta(\cdots) \frac{x_1^{-1+\epsilon}}{\big(m^2 x_1 + (m^2+M^2-s)x_2\big)^{1+\epsilon}} \\ &= -M^{-2}\Big(\frac{m^2}{\mu^2}\Big)^{-2\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon)(1-x) \\ &= \frac{1-x}{M^2}\Bigg(-\frac{1}{\epsilon} + \log\frac{m^2}{\mu^2}\Bigg)\,. \end{aligned} \tag{91}$$

  Here, we once again see that we have lost some complexity since the $\mathcal{U}$ polynomial is trivial.

Both I_c and I_h are divergent but their sum is finite as was the original integral. However, $I_h \propto (M^2/\mu^2)^{-\epsilon}$ and $I_c \propto (m^2/\mu^2)^{-\epsilon}$ so that $\log(M^2/m^2)$ remains

$$I_c + I_h = -\frac{1-x}{xM^2}\Big(x\log\frac{M^2}{m^2} + (1-2x)\log(1-x)\Big)\,. \tag{92}$$

This is a common feature in the method of regions: all regions are separately more divergent than the total result which can introduce new logarithms.

4.2 Parameter space

While the method of region is doubtlessly an invaluable tool, especially when we want to connect its regions to effective theories. However, if all we want to do is calculate integrals without having to bother with a physical intuition, it can be unwieldy, especially since some regions are not visible for a given momentum routing. It turns out we can formulate the method of regions at the level of the Feynman parameters which is much more general and allows for easier automation [24, 25].

Since an important aspect of the method of region is scalelessness, we need to formalise this at the level of the Feynman parameters.

Lemma 41 (Scaleless integrals)**.** We call an integral scaleless if rescaling the loop momentum and/or the external momenta results in a global factor. These integrals vanish in dimensional regularisation. For example,

$$I = \int [\mathrm{d}k] \frac{1}{\left[k^2\right]^n} \to \int [\mathrm{d}(\alpha k)] \frac{1}{\left[(\alpha k)^2\right]^n} = \alpha^{d-2n} I = 0\,. \tag{93}$$

In the language of Feynman parameters, the integral vanishes if we can rescale a strict subset of parameters

$$\begin{aligned} \mathcal{U}(x_1, \cdots, \alpha x_j, \cdots, x_n) &= \alpha^u \mathcal{U}(x_1, \cdots, x_j, \cdots, x_n) \\ \mathcal{F}(x_1, \cdots, \alpha x_j, \cdots, x_n) &= \alpha^f \mathcal{F}(x_1, \cdots, x_j, \cdots, x_n)\,, \end{aligned} \tag{94}$$

for some scaling dimensions u and f. For simplicity we only consider $\mathcal{UF} \to \alpha^{u+f}\mathcal{UF}$.

Theorem 42 (The method of regions in parameter space)**.** To expand a parameter integral, we multiply the small kinematic invariants with appropriate power of λ in $\mathcal{F}$. A region is defined as a $(n+1)$-dimensional vector $\vec{v} = (1, v_1, \cdots, v_n)$ where $x_i \sim \lambda^{v_i}$. We can relate $\vec{v}$ to the complex hull of the point cloud that is spanned by $\mathcal{U}$ and $\mathcal{F}$ (see proof and Example 43 for details). This is implemented in public codes such as `asy` [24, 25] or `pySecDec` [26].

Example 43 (Example in parameter space)**.** For the example (86) we have $\vec{v}_h = (1, 0, 0)$ and $\vec{v}_c = (1, 0, -1)$. We have hence

$$I_h = -\Gamma(1-\epsilon)\Gamma(1+\epsilon) \int \mathrm{d}x_1 \mathrm{d}x_2 \delta(\cdots) x_2 \frac{(x_1+x_2)^{-1+2\epsilon}}{\big(M^2 x_1^2 + (M^2 - s) x_1\, x_2\big)^{1+\epsilon}}\,, \tag{95}$$

$$I_c = -\Gamma(1-\epsilon)\Gamma(1+\epsilon) \int \mathrm{d}x_1 \mathrm{d}x_2 \delta(\cdots) \frac{x_2^{2\epsilon}}{\big(m^2 x_2^2 + (M^2 - s) x_1\, x_2\big)^{1+\epsilon}}\,. \tag{96}$$

The integration of the Feynman parameters is now trivial again as above.

Proof of Theorem 42. Consider a term in the Lee-Pomeransky polynomial $\mathcal{F} + \mathcal{U}$ (cf. (28)), it will have the structure

$$\lambda^{r_0} x_1^{r_1} \cdots x_n^{r_n} \equiv \vec{r} = (r_0, r_1, \ldots, r_n)\,. \tag{97}$$

We view the terms as a point cloud $C_{\mathcal{F}+\mathcal{U}}$ in an $(n+1)$ dimensional space that we sometimes split as $C_{\mathcal{F}+\mathcal{U}} = C_{\mathcal{F}} \cup C_{\mathcal{U}}$. $\mathcal{F}$ is homogeneous, i.e. $r_1 + \cdots + r_n = \ell + 1$, meaning that all terms in $C_{\mathcal{F}}$ live on an n dimensional hyperplane. Were the dimensionality lower than n, we could factor out a Feynman parameter, making the integral scaleless.

For a given region, this term will scale as

$$\lambda^{r_0} \big(x_1 \lambda^{v_1}\big)^{r_1} \cdots \big(x_n \lambda^{v_n}\big)^{r_n} \sim \lambda^{r_0 + v_1 r_1 + \cdots + v_n r_n} = \lambda^{\vec{v}\cdot\vec{r}}\,. \tag{98}$$

After expanding $\mathcal{F}$ and $\mathcal{U}$ in λ, all remaining terms have to have the same scaling since otherwise the expansion would not have been complete. In other words, the remaining terms belong to a hyperplane orthogonal to $\vec{v}$. Points that are above this hyperplane are more suppressed in λ since they have larger r_0, the power of λ.

If the dimensionality of this hyperplane is lower than n, the integral is scaleless. Thus we are looking for facets of the envelope of the point cloud $C_{\mathcal{F}+\mathcal{U}}$. The $\vec{v}$ are the (inwards facing) normal vectors of these facets with $v_1 > 0$. This automatically selects only the bottom facets.

□

Example 44 (Construction of $\vec{v}$)**.** For the example (86) we have

$$\begin{aligned} \mathcal{F} &= M^2 x_1^2 + M^2 x_1 x_2 - s x_1 x_2 + m^2 x_2^2 \lambda\,, \\ \mathcal{U} &= x_1 + x_2\,. \end{aligned} \tag{99}$$

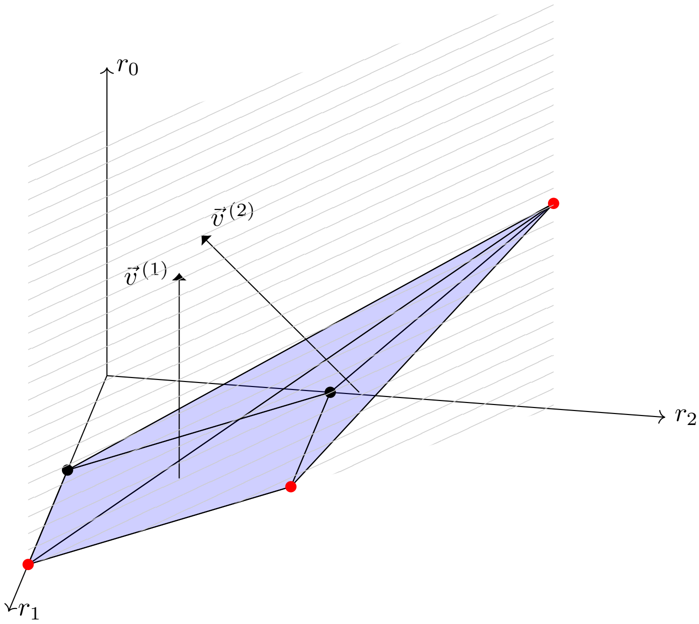


Figure 8: The point clouds $C_{\mathcal{F}}$ in red and $C_{\mathcal{U}}$ in black and their convex hull in blue. The grey shaded region corresponds to the requirement that $\mathcal{F}$ is homogeneous, i.e. $r_1 + r_2 = \ell + 1$. An interactive version of this plot can be found on the course website.

This results in the point cloud

$$\begin{aligned} C_{\mathcal{F}} &= \Big\{(0,2,0),(0,1,1),(0,1,1),(1,0,2)\Big\} \\ C_{\mathcal{U}} &= \Big\{(0,1,0),(0,0,1)\Big\} \end{aligned} \tag{100}$$

as shown in Figure 8. Next, we construct the convex hull of $C_{\mathcal{F}+\mathcal{U}} = C_{\mathcal{F}} \cup C_{\mathcal{U}}$. It has five facets of which we only care about two, those spanned by

$$F_1 = \begin{pmatrix} \vec{r}^{(1)} \\ \vec{r}^{(2)} \\ \vec{r}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} \vec{r}^{(1)} \\ \vec{r}^{(2)} \\ \vec{r}^{(3)} \\ \vec{r}^{(4)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix} . \tag{101}$$

We can find the normal vectors $\vec{v}^{(1)}$ and $\vec{v}^{(2)}$ by finding the kernel of these matrices as

$$\vec{v}^{(1)} = \big(1, 0, -1\big) \quad \text{and} \quad \vec{v}^{(2)} = \big(1, 0, 0\big) \tag{102}$$

Here, $\vec{v}^{(2)}$ corresponds to the hard region and $\vec{v}^{(1)}$ to the collinear region. We do not, for example, care about this facet

$$F_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \tag{103}$$

which has $\vec{v}^{(3)} = (0, -1, -1)$.

Lemma 45 (Lee-Pomeransky vs. Feynman representation)**.** The regions obtained from the Lee-Pomeransky representation $C_{\mathcal{F}+\mathcal{U}}$ are identical to those of the Feynman representation $C_{\mathcal{U}\mathcal{F}}$ since their lower facets have a one-to-one correspondence [27]. Using this, we can now consider $n = 3$ integrals and still visualise the result. We do this by first projecting out the λ dimension in $C_{\mathcal{F}+\mathcal{U}}$ and then constructing $C_{\mathcal{U}\mathcal{F}}$. Because $C_{\mathcal{U}\mathcal{F}}$ lies in a single hyperplane, we can reintroduce the λ direction as orthogonal to that plane.

Proof. We can construct a surface $C'_{\mathcal{UF}}$ by finding the intersection between all vertices in $C_{\mathcal{U}}$ and $C_{\mathcal{F}}$. This surface now corresponds to $C_{\mathcal{UF}}$ up to a rescaling. There is a statement from alebraic geometry called Cayley's trick states that the convex hulls of $C'_{\mathcal{UF}}$ and $C_{\mathcal{U}+\mathcal{F}}$ are identical. □

Lemma 46 (Connecting parameter and momentum regions)**.** For a given momentum region, we find the (inverse) region vector $-\vec{v}$ by considering how each propagator scales. Consider for example the collinear region of Example 40

$$\mathcal{P}_1 = (k+p)^2 - M^2 = \lambda^0 \Big[2k_- \cdot p_+\Big] + \mathcal{O}(\lambda)\,, \tag{104a}$$

$$\mathcal{P}_2 = (k-q)^2 - m^2 = \lambda^2 \Big[k^2 - 2k\cdot q\Big]\,. \tag{104b}$$

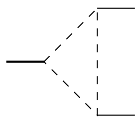
This region has $\vec{v}^{\,(c)} = (0,-2)$.

Proof. Consider the denominator after Feynman parametrisation (12)

$$D = \mathcal{P}_1 x_1 + ... + \mathcal{P}_t x_t \tag{105}$$

The expansion is only complete if there is no scale seperation between the different terms. This means that if $\mathcal{P}_i$ scales as λ^{-v_i}, x_i needs to scale as λ^{v_i}. □

Example 47 (Massless Sudakov form factor [28])**.** Consider the following integral

$I =$ 

$$= \int \frac{[\mathrm{d}k]}{k^2\ (k-p)^2\ (k-q)^2}\,, \tag{106}$$

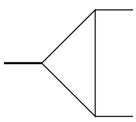
with $p^2 = q^2 = m^2 \ll (p+q)^2 = s$. We can easily construct

$$\mathcal{U} = x_1 + x_2 + x_3 \qquad \text{and} \qquad \mathcal{F} = s x_1 x_2 - m^2\lambda(x_1x_3 + x_2x_3) \tag{107}$$

to find the point clouds $C_{\mathcal{U}}$ and $C_{\mathcal{F}}$. The projection along the λ direction is shown in Figure 9a with the point cloud for $C_{\mathcal{UF}}$ drawn in between. The resulting hexagon for $C_{\mathcal{UF}}$ can now be rotated and again stretched out in the λ direction as shown in Figure 9b. This allows us now to identify the upward pointing facets and identify four regions

$$\begin{pmatrix}\vec{v}^{\,(1)}\\ \vec{v}^{\,(2)}\\ \vec{v}^{\,(3)}\\ \vec{v}^{\,(4)}\end{pmatrix} = \begin{pmatrix}0&0&0\\-2&0&-2\\-2&-2&0\\-4&-2&-2\end{pmatrix} \equiv \begin{pmatrix}\text{hard}\\ \text{collinear to } q\\ \text{collinear to } p\\ \text{soft}\end{pmatrix}\,. \tag{108}$$

Example 48 (Massive Sudakov form factor)**.** Consider a slightly modified integral

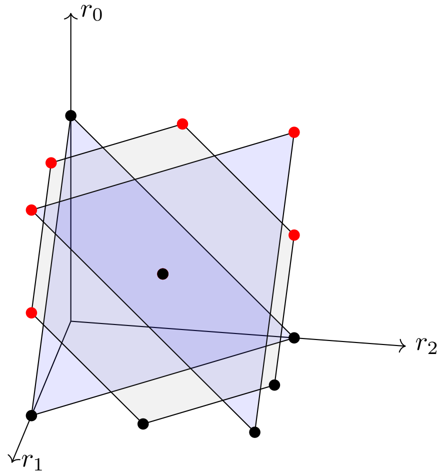
$I' =$ 

$$= \int \frac{[\mathrm{d}k]}{(k^2-m^2)\ ((k-p)^2-m^2)\ ((k-q)^2-m^2)}\,, \tag{109}$$

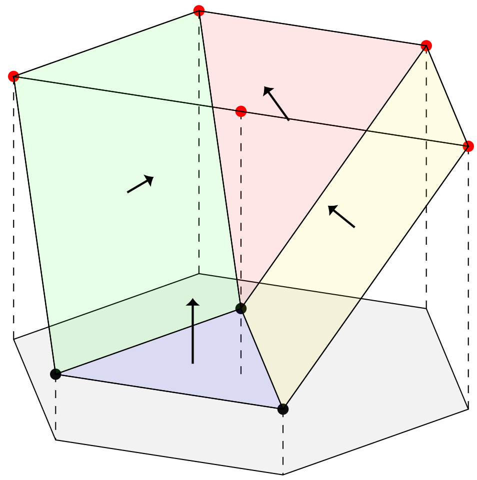
with $p^2 = q^2 = m^2 \ll (p+q)^2 = s$. We now get $\mathcal{U}' = \mathcal{U}$ and $\mathcal{F}' = \mathcal{F} + \mathcal{U}(x_1+x_2+x_3)m^2\lambda^2$. Constructing again the convex hull for $C_{\mathcal{UF}}$ we note that it now has one face less. This also translates to a missing soft region

$$\begin{pmatrix}\vec{v}^{\,(1)}\\ \vec{v}^{\,(2)}\\ \vec{v}^{\,(3)}\end{pmatrix} = \begin{pmatrix}0&0&0\\-2&0&-2\\-2&-2&0\end{pmatrix} \equiv \begin{pmatrix}\text{hard}\\ \text{collinear to } q\\ \text{collinear to } p\end{pmatrix}\,. \tag{110}$$

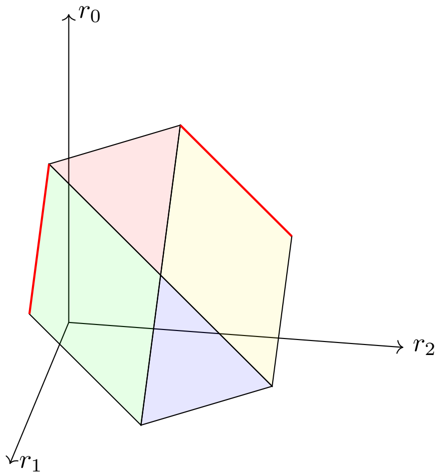
Lemma 49 (Singularities and the convex hull)**.** Singularities of an integral arise from facets of the convex hull that lie on a plane that also contains $\vec{0}$ [26]. We can now distinguish two cases



(a) $C_{\mathcal{U}}$, $C_{\mathcal{F}}$ and $C_{\mathcal{UF}}$ in grey

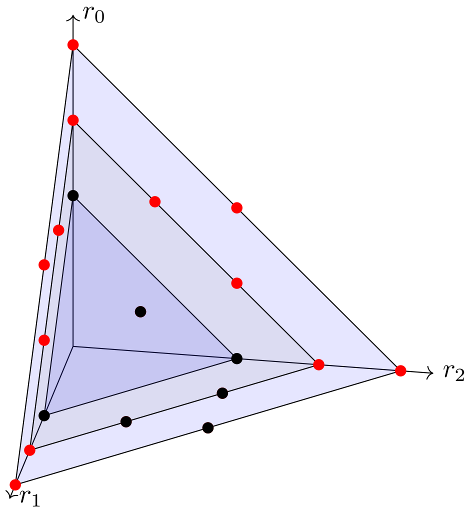


(b) $C_{\mathcal{UF}}$ in the rotated frame with the convex hull

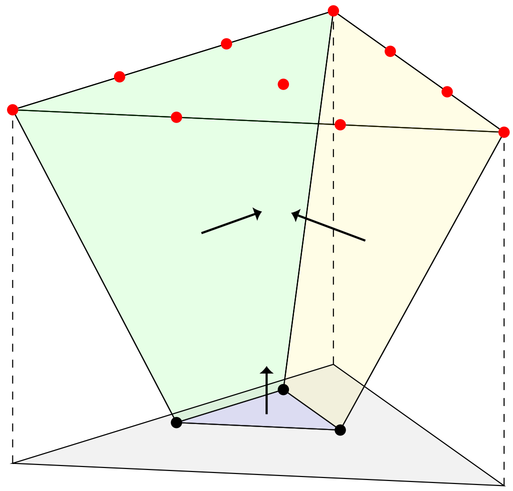


(c) Regions identified on $C_{\mathcal{UF}}$ with non-affine facets highlighted in red

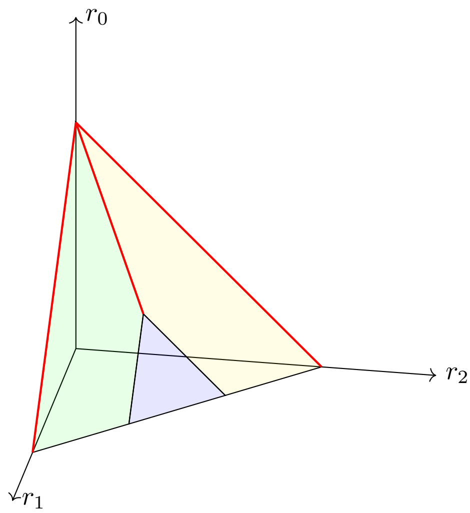
Figure 9: The geometric construction of the massless Sudakov form factor. For explanation, see Example 47



(a) $C_{\mathcal{U}}$, $C_{\mathcal{F}}$ and $C_{\mathcal{UF}}$ in grey



(b) $C_{\mathcal{UF}}$ in the rotated frame with the convex hull



(c) Regions identified on $C_{\mathcal{UF}}$ with non-affine facets highlighted in red

Figure 10: The geometric construction of the massive Sudakov form factor. For explanation, see Example 48

- *outer facets* that touch at most one region are regulated dimensionally and exist independent of the method of regions,
- *inner facets* that touch two regions are not regulated dimensionally and require an analytic regulator, i.e. a shift of the propagator powers.

These are highlighted in red Figures 9c and 10c. It is clear that both have two outer facets but the latter also has in inner facet which means the regions are not regulated dimensionally.

Proof. In this proof, we follow [26, 29]. In the Lee-Pomeransky representation, the most general case we need to consider is

$$I = \int \Big(\prod_{i=1}^{t} \mathrm{d}x_i \ x_i^{\alpha_i - 1}\Big)\Big(\sum_k c_k \vec{x}^{\ \vec{r}_k}\Big)^{-d/2} . \tag{111}$$

One can show that the integration region $\mathbb{R}^t$ can be decomposed into into simplicial cones defined by rays $\vec{n}_F$

$$\sigma = \Big\{ \sum_{F\in\sigma} a_F \vec{n}_F \ : \ a_F > 0 \Big\} . \tag{112}$$

The rays $\vec{n}_F$ are the normal vectors of a facet F of the convex hulls discussed above. The facets themselves can be described in Hessian normal form by a normal vector $\vec{n}$ and an offset a as

$$F = \{\vec{y} \ : \ \vec{n}_F \cdot \vec{y} + a_F = 0\} . \tag{113}$$

By considering all t-dimensional cones σ we can reconstruct the full integration region $\mathbb{R}^t$. Transforming from the cartesian coordinates $\{\vec{e}_i\}$ to the local coordinates of the facets we have

$$x_i = \prod_{F\in\sigma} y^{\vec{n}_F \cdot e_i} , \tag{114}$$

for each cone. We now have

$$I = \sum_\sigma |\sigma| \prod_{F\in\sigma} \int \mathrm{d}y_F \ y_F^{\vec{n}_F\cdot\vec{\alpha}-1} \Big(\sum_k c_k \prod_{F\in\sigma} y_F^{\vec{n}_F\cdot\vec{r}_k}\Big)^{-d/2} , \tag{115}$$

with the Jacobian $|\sigma|$. To guarantee that the complicated monomial has positive exponents, we factor our $y_F^{-a_F}$.

$$I = \sum_\sigma |\sigma| \prod_{F\in\sigma} \int \frac{\mathrm{d}y_F}{y_F} y_F^{\vec{n}_F\cdot\vec{\alpha}+\frac{d}{2}a_F} \underbrace{\Big(\sum_k c_k \prod_{F\in\sigma} y_F^{\vec{n}_F\cdot\vec{r}_k + a_F}\Big)^{-d/2}}_{g(y)} . \tag{116}$$

Since the $\vec{r}_k \in F$, the g part is now free of singularities. Hence, for I to be finite, we require

$$\vec{n}_F \cdot \vec{\alpha} + \frac{d}{2} a_F > 0 . \tag{117}$$

If $a_F \neq 0$, the integral is regulated dimensionally. The analytic regulator now sets some $\alpha_i \to \alpha_i \pm \eta$ with $\eta > 0$ such that the new $\vec{\alpha}$ is no longer orthogonal to any $\vec{n}_F$. □

5 Mellin Barnes integration

When calculating loop integrals, we are often faced with having to integrate polynomials to some non-integer power. Since we can (essentially) only integrate using the definition of the Γ function[4] the most complicated trivially solvable integral is

$$\int_0^\infty \mathrm{d}x \ x^\alpha (a+bx)^\beta = \frac{\Gamma(1+\alpha)\Gamma(-1-\alpha-\beta)}{\Gamma(-\beta)} \frac{a^{1+\alpha+\beta}}{b^{1+\alpha}} . \tag{8}$$

It goes without saying that most real-world integrals are more complicated than this.

[4]Technically, we can also solve integrals of the form $\int \mathrm{d}x, x^a(1+x)^b(z+x)^c$ using the methods presented in this section. Hence this integral is just a shortcut to what we show here.

5.1 The Mellin Barnes theorem

Observation 50 (Heavy-quark form factor)**.** Consider once again the following one-loop integral

$$I(a) = \int [\mathrm{d}k] \; \frac{1}{\big[k^2\big]^a \big[k^2 + 2k\cdot p\big]\big[k^2 - 2k\cdot q\big]} \tag{118}$$

with $p^2 = q^2 = m^2$, $(p+q)^2 = s$ and some integer a. After Feynman parametrisation we find (see eg. Observation 7)

$$I(a) = (-1)^a \frac{\Gamma(1-\epsilon)\Gamma(a+\epsilon)}{\Gamma(a)} \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \; \delta(\cdots) x_1^{a-1} \frac{\big(x_1 + x_2 + x_3\big)^{-2+a+2\epsilon}}{\big(m^2\, x_2^2 + (2m^2 - s)\, x_2 x_3 + m^2\, x_3^2\big)^{a+\epsilon}} \,. \tag{119}$$

We can trivially solve the x_1 integral using (8)

$$I(a) = (-1)^a \frac{\Gamma(1-\epsilon)\Gamma(2-2a-2\epsilon)\Gamma(a+\epsilon)}{\Gamma(2-a-2\epsilon)} \int \mathrm{d}x_2\mathrm{d}x_3 \; \delta(\cdots) \frac{\big(x_2 + x_3\big)^{-2+2a+2\epsilon}}{\big(m^2\,(x_2+x_3)^2 - s\, x_2 x_3\big)^{a+\epsilon}} \,, \tag{120}$$

but now are stuck. Obviously we cannot just use (8) to solve this integral. However, if we could somehow split the $\mathcal{F}$ polynomial into the terms $\propto m^2$ and those $\propto s$, we might stand a chance. To do this, we note the Mellin-Barnes theorem.

Theorem 51 (Mellin-Barnes split)**.** We can split an arbitrary polynomial into two factors at the cost of an integration. You can think of this akin to an inverse Feynman parametrisation

$$\frac{1}{(A+B)^\lambda} = \frac{1}{2\pi\mathrm{i}} \frac{1}{\Gamma(\lambda)} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \; \Gamma(\lambda+z)\Gamma(-z) A^z B^{-z-\lambda} \,. \tag{121}$$

We ignore the factor $1/(2\pi\mathrm{i})$ as it will cancel with the one from the residue theorem when we eventually calculate the contour integral. The contour in question is chosen to separate the poles from the left and right Γ functions. We will soon see how this works in practise.

Crucially, we are not allowed to write eg.

$$\frac{\Gamma(1+z)\Gamma(-z)}{\Gamma(z)} = -\Gamma(1-z) \tag{122}$$

since it would confuse left and right poles.

Definition 52 (Left and right Γ functions)**.** $\Gamma(z)$ functions has infinitely many poles at $z = -n$. We hence call Γ functions of the form $\Gamma(\cdots - z)$ right Γ functions as their poles go right towards $+\infty$. Similarly, $\Gamma(\cdots + z)$ are left Γ functions.

Observation 53 (Heavy-quark form factor, Pt. 2)**.** Using (121), we write I as

$$\begin{aligned} I(a) = (-1)^a (m^2)^{-a-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-2a-2\epsilon)}{\Gamma(2-a-2\epsilon)} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}z \Big(\frac{-s}{m^2}\Big)^z \Gamma(-z)\Gamma(a+\epsilon+z) \\ \int \mathrm{d}x_2\mathrm{d}x_3 \; \delta(\cdots) \frac{x_2^z x_3^z}{(x_2+x_3)^{2+2z}} \,. \end{aligned} \tag{123}$$

Now we can use (8) to solve this integral, using for example $\delta(\cdots) = \delta(1-x_3)$

$$\int \mathrm{d}x_2\mathrm{d}x_3 \; \delta(\cdots) \frac{x_2^z x_3^z}{(x_2+x_3)^{2+2z}} = \frac{\Gamma(1+z)^2}{\Gamma(2+2z)} \,. \tag{124}$$

Therefore for I

$$I(a) = (-1)^a (m^2)^{-a-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-2a-2\epsilon)}{\Gamma(2-a-2\epsilon)} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}z \Big(\frac{-s}{m^2}\Big)^z \frac{\Gamma(-z)\Gamma(a+\epsilon+z)\Gamma(1+z)^2}{\Gamma(2+2z)} \,. \tag{125}$$

```
GammaResidue[expr_,{z_,pole_}]:=Assuming[
  Element[n,Integers] && n>=0,
  Residue[expr,{z,pole}]
]/.n!-> Gamma[n+1]
```

Listing 11: Mathematica calculation of the residue

5.2 Solving Mellin Barnes integrals using the residue theorem

Theorem 54 (Residue theorem for MB)**.** Assuming the MB integrand falls quickly enough, i.e. the arc C' vanishes (see eg. Figure 12), we can solve a single MB integral using the residue theorem. Consider

$$I = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}z\ \underbrace{x^z \left(\prod_{i=1}^{n_L} \Gamma(a_i+z)^{c_i}\right)\left(\prod_{i=1}^{n_R} \Gamma(b_i-z)^{d_i}\right)\big(\cdots\big)^{-1}}_{f(z)} \tag{126}$$

with $c_i, d_i > 0$. Then

$$I = \sum_{i=1}^{n_L}\sum_{n=0}^{\infty} \operatorname*{res}_{-a-n} f(z) = \sum_{i=1}^{n_R}\sum_{n=0}^{\infty} \operatorname*{res}_{b+n} f(z)\,. \tag{127}$$

Lemma 55 (Residue formulas)**.** We remember that if $f(z)$ has a pole of order n at c,

$$\operatorname*{res}_{c} f(z) = \frac{1}{(n-1)!}\lim_{z\to c}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\Big[(z-c)^n f(z)\Big]\,. \tag{128}$$

If $f(z) = g(z)h(z)$ and h is free of poles at c, this means

$$\operatorname*{res}_{c} f(z) = h(c)\operatorname*{res}_{c} g(z)\,. \tag{129}$$

Finally,

$$\operatorname*{res}_{-n}\Gamma(z) = \frac{(-1)^n}{n!}\,, \tag{130}$$

$$\operatorname*{res}_{-n}\Gamma(z)^2 = \frac{2}{(n!)^2}\psi(n+1)\,, \tag{131}$$

$$\operatorname*{res}_{-n}\Gamma(z)^3 = \frac{(-1)^n}{2(n!)^4}\Big[\pi^2 + 9\psi(n+1)^2 - 3\psi'(n+1)\Big]\,, \tag{132}$$

$$\operatorname*{res}_{-n}\psi(z) = -1\,. \tag{133}$$

Here we have also included the derivative of the Γ function, the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$. Note that these rules can be implemented in Mathematica with an `Assumption` as shown in Listing 11.

Observation 56 (Heavy-quark form factor, Pt. 3)**.** For simplicity we close the contour to the right as shown in Figure 12 for $a=1$. We have one residue to calculate since $n_R = 1$

$$\begin{aligned} I(a) &= (-1)^a (m^2)^{-a-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-2a-2\epsilon)}{\Gamma(2-a-2\epsilon)}\sum_{n=0}^{\infty}\operatorname*{res}_{n}\left[\left(\frac{-s}{m^2}\right)^z\frac{\Gamma(-z)\Gamma(a+\epsilon+z)\Gamma(1+z)^2}{\Gamma(2+2z)}\right] \\ &= (-1)^{a+1}(m^2)^{-a-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-2a-2\epsilon)}{\Gamma(2-a-2\epsilon)}\sum_{n=0}^{\infty}\left(\frac{s}{m^2}\right)^n\frac{\Gamma(1+n)\Gamma(a+n+\epsilon)}{\Gamma(2+2n)}\,. \end{aligned} \tag{134}$$

We can solve the remaining sum and find

$$I(a) = (-1)^{a+1}(m^2)^{-a-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-2a-2\epsilon)}{\Gamma(2-a-2\epsilon)}\Gamma(1+\epsilon)\,{}_2F_1\left[\begin{matrix}1,\ 1+\epsilon\\ \frac{3}{2}\end{matrix};\frac{s}{4m^2}\right]. \tag{135}$$

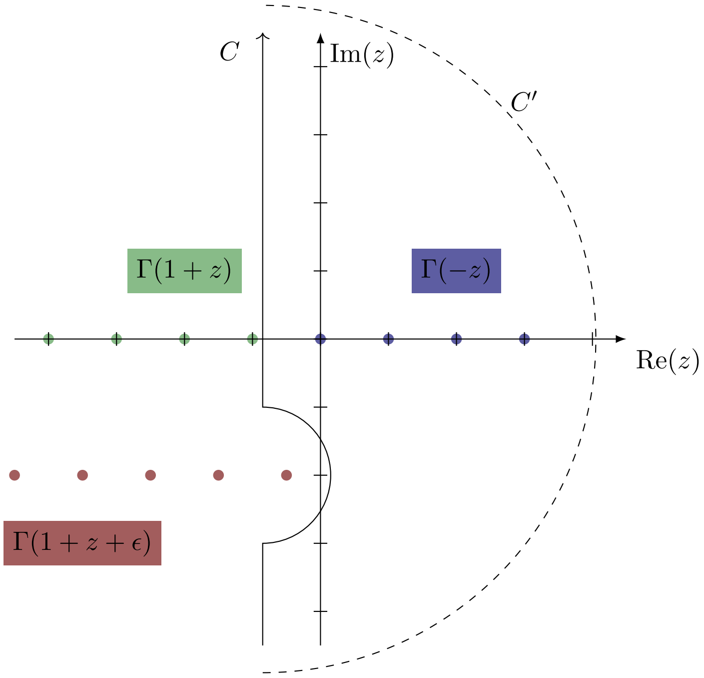


Figure 12: The contour for the MB integral (125) for $a = 1$

Definition 57 (Hypergeometric function)**.** We define the hypergeometric function as

$$ {}_pF_q\left[\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{136} $$

with the Pochhammer symbol $a_n = \Gamma(a+n)/\Gamma(n)$ and the factorial $n! = \Gamma(1+n)$. For certain values of a_i and b_i, these functions can be expanded in ϵ using `HypExp` [23].

5.3 Solving Mellin Barnes integrals using Barnes lemma

Observation 58 (Resolving singularities)**.** Let us consider the special case of $a = 0$ (ignoring the fact that there are easier ways to obtain this result)

$$ I(0) = (m^2)^{-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-2\epsilon)}{\Gamma(2-2\epsilon)} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}z \left(\frac{-s}{m^2}\right)^z \frac{\Gamma(-z)\Gamma(\epsilon+z)\Gamma(1+z)^2}{\Gamma(2+2z)}\,. \tag{137} $$

Setting $\epsilon \to 0$ is obviously not possible since the left pole from $\Gamma(\epsilon + z)$ would collide with the right pole from $\Gamma(-z)$, meaning the contour would have to cross a pole. However, if we moved the contour as shown in Figure 13 then we can set $\epsilon \to 0$. By crossing a pole with the contour we have to explicitly add the residue of the pole

$$ I(0) = (m^2)^{-\epsilon}\Gamma(1-\epsilon)\left[\int_{-\frac{1}{2}-\mathrm{i}\infty}^{-\frac{1}{2}+\mathrm{i}\infty} \mathrm{d}z \left(\frac{-s}{m^2}\right)^z \frac{\Gamma(-z)\Gamma(\epsilon+z)\Gamma(1+z)^2}{\Gamma(2+2z)} - \underbrace{\operatorname*{res}_0\Big[\cdots\Big]}_{-\Gamma(\epsilon)}\right] \tag{138} $$

$$ = \frac{1}{\epsilon} - \log m^2 + \int_{-\frac{1}{2}-\mathrm{i}\infty}^{-\frac{1}{2}+\mathrm{i}\infty} \mathrm{d}z \left(\frac{-s}{m^2}\right)^z \frac{\Gamma(-z)\Gamma(+z)\Gamma(1+z)^2}{\Gamma(2+2z)} + \mathcal{O}(\epsilon)\,. \tag{139} $$

At this stage we may expand in ϵ to whatever order we require.

Note that we can also numerically integrate (139).

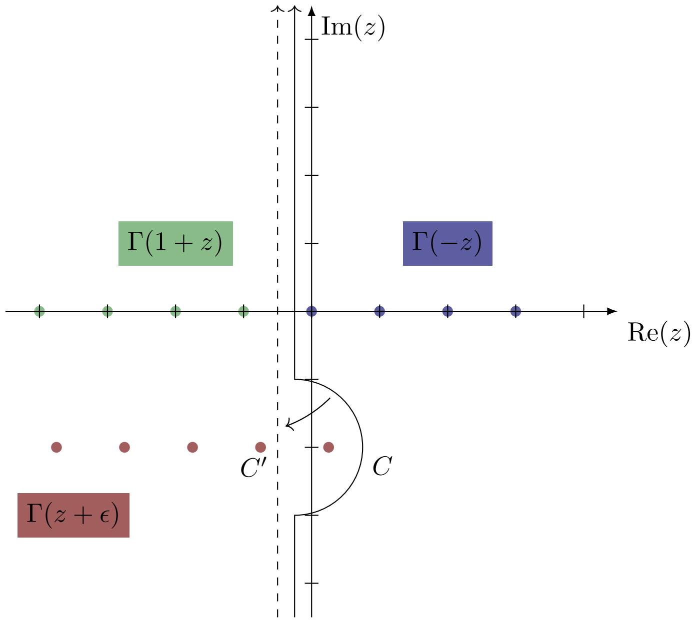


Figure 13: The contour for the MB integral (125) for $a = 0$

Observation 59 (Slightly contrived box)**.** Consider this (slightly contrived) example

$$I(a) = \int [\mathrm{d}k] \frac{1}{\left[k^2\right]\left[(k-p_1)^2\right]\left[(k+p_2)^2\right]\left[(k-p_1+p_4)^2\right]}\,, \tag{140}$$

with $p_1^2 = p_3^2 = (p_1+p_2)^2 = (p_1-p_3)^2 = s$ and $p_2^2 = p_4^2 = 0$. This example could for example occur when calculating the boundary condition for some differential equations of Feynman integrals. There have usually much simpler and often unphysical kinematics and are then evolved to the physics situation. We have

$$I = (-s)^{-2-\epsilon}\Gamma(1-\epsilon)\Gamma(2+\epsilon)\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)\frac{\big(x_1+x_2+x_3+x_4\big)^{2\epsilon}}{\big(x_1\,x_2+x_3\,(x_2+x_4)\big)^{2+\epsilon}}\,. \tag{141}$$

We split the x_3 term using Mellin Barnes

$$\begin{aligned} I = (-s)^{-2-\epsilon}\Gamma(1-\epsilon)\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z\Gamma(-z)\Gamma(2+\epsilon+z)\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)& \\ x_1^{-2-z-\epsilon}\;x_2^{-2-z-\epsilon}\;x_3^{z}\;(x_3+x_4)^z\;(x_1+x_2+x_3+x_4)^{2\epsilon}\,.& \end{aligned} \tag{142}$$

We can now obtain

$$I = (-s)^{-2-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(-2\epsilon)}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z\frac{\Gamma(-z)\Gamma(1+z)\Gamma(-1-\epsilon-z)^2\Gamma(2+\epsilon+z)}{\Gamma(-\epsilon-z)}\,. \tag{143}$$

We could obviously solve this with the residue theorem but we have two left poles or two right poles, one of them involving $\Gamma(\cdots - z)^2$. Let us try to expand in ϵ by resolving the singularities

$$\begin{aligned} I &= (-s)^{-2-\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(-2\epsilon)}\left(\int_{-\frac12-\mathrm{i}\infty}^{-\frac12+\mathrm{i}\infty}\mathrm{d}z\frac{\Gamma(-z)\Gamma(1+z)\Gamma(-1-\epsilon-z)^2\Gamma(2+\epsilon+z)}{\Gamma(-\epsilon-z)} - \operatorname*{res}_{-1-\epsilon}\Big(\cdots\Big)\right) \\ &= 2(-s)^{-2-\epsilon}\Bigg[\frac{1}{\epsilon^2} - 2\zeta_2 + \int_{-\frac12-\mathrm{i}\infty}^{-\frac12+\mathrm{i}\infty}\mathrm{d}z\Gamma(1+z)\Gamma(2+z)\Gamma(-1-z)^2 \\ &\qquad - \left(2\zeta_3 + \int_{-\frac12-\mathrm{i}\infty}^{-\frac12+\mathrm{i}\infty}\mathrm{d}z\Gamma(1+z)\Gamma(2+z)\Gamma(-1-z)^2\Big(\psi(-z) - 2\psi(-1-z) + \psi(2+z)\Big)\right)\epsilon\Bigg]\,. \end{aligned} \tag{144}$$

Theorem 60 (Barnes Lemma)**.** We have two Barnes Lemma that can be used to calculate integrals

$$\int_{-i\infty}^{+i\infty} dz\ \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z) = \frac{\Gamma(\lambda_{13})\Gamma(\lambda_{14})\Gamma(\lambda_{23})\Gamma(\lambda_{24})}{\Gamma(\lambda_{1234})}\,, \tag{145}$$

$$\int_{-i\infty}^{+i\infty} dz\ \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_{12345}+z)} = \frac{\Gamma(\lambda_{14})\Gamma(\lambda_{24})\Gamma(\lambda_{34})\Gamma(\lambda_{15})\Gamma(\lambda_{25})\Gamma(\lambda_{35})}{\Gamma(\lambda_{1245})\Gamma(\lambda_{1345})\Gamma(\lambda_{2345})}\,, \tag{146}$$

with $\lambda_{12} = \lambda_1 + \lambda_2$ etc. The Barnes Lemma are particularly useful when dealing with multiple MB integrals. Further, they can be used to generate further identities. For example, taking the derivative of (145) w.r.t. λ_1

$$\begin{aligned}\int_{-i\infty}^{+i\infty} dz\ \Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z)\psi(\lambda_1+z) &= \frac{\Gamma(\lambda_{13})\Gamma(\lambda_{14})\Gamma(\lambda_{23})\Gamma(\lambda_{24})}{\Gamma(\lambda_{1234})} \\ &\quad\times\Big[\psi(\lambda_{13})+\psi(\lambda_{14})-\psi(\lambda_{1234})\Big]\,.\end{aligned} \tag{147}$$

Observation 61 (Slightly contrived box, Pt. 2)**.** We can now calculate the $\mathcal{O}(\epsilon^0)$ term of (144), and by taking the derivative w.r.t. the λ_i

$$I = 2(-s)^{-2-\epsilon}\left[\frac{1}{\epsilon^2} - \zeta_2 - \left(-\zeta_3 + \gamma_E\zeta_2 + \int_{-\frac12-i\infty}^{-\frac12+i\infty} dz\Gamma(1+z)\Gamma(2+z)\Gamma(-1-z)^2\psi(-z)\right)\epsilon\right]. \tag{148}$$

It is possible to write $\psi(-z)$ as $\psi(1+z)$ and again use the Barnes Lemmas. Alternatively, we can note that

$$\Gamma(-1-z)\Gamma(2+z) = -\frac{1}{1+z}\Gamma(-z)\Gamma(2+z) = -\Gamma(1+z)\Gamma(-z)\,. \tag{149}$$

We are now allowed do this, since we have specified the contour explicitly and no longer need to distiguish left and right poles. We write

$$\begin{aligned}\int_{-\frac12-i\infty}^{-\frac12+i\infty} dz\Gamma(1+z)\Gamma(2+z)\Gamma(-1-z)^2\psi(-z) &= -\int_{-\frac12-i\infty}^{-\frac12+i\infty} dz\Gamma(1+z)^2\Gamma(-z)\Gamma(-1-z)\psi(-z) \\ &= -2\zeta_3 - \gamma_E\zeta_2\,.\end{aligned} \tag{150}$$

With this

$$I = 2(-s)^{-2-\epsilon}\left[\frac{1}{\epsilon^2} - \zeta_2 - 3\zeta_3\epsilon\right]. \tag{151}$$

Naturally, doing this manually quickly becomes annoying, the higher we want to go in ϵ.

Observation 62 (Strategies)**.** In the example above, the integral has trivial dependence on external kinematics. In general, this means that it can be written as $I = (Q^2)^{\ell\epsilon} f(\epsilon)$ where f is some function that only depends on the dimensionality of spacetime. Using the methods described in Observation 58, we are able to expand f around $\epsilon = 0$ and calculate the remaining integrals numerically. We will shortly see a particularly clever way to exploit this.

If that were not the case, we have to explicitly sum the residues. Mathematica is capable of doing this in a limited number of cases and the `FORM` code `XSummer` [30] covers many more. However, all of this works only if there is only a single integration left. Finding a series expansion for multiple MB integrals is highly non-trivial and subject to active research (eg. [31]).

Theorem 63 (PSLQ algorithm)**.** Given some real (notionally transcendental) numbers a_i, we can find rational numbers r_i such that

$$r_0\, a_0 + r_1\, a_1 + \cdots + r_n\, a_n = 0 \tag{152}$$

using black magic, also known as number theory. An implementation of this algorithm can be found with these lecture notes.

Observation 64 (Using PSLQ)**.** If we can guess a basis of transcendetal numbers such as

$$\{a_i\}_{i=1,\cdots,n} = \{1, \zeta_2, \zeta_3, \zeta_4, \cdots, \log 2, \log 2\zeta_2, \cdots\}\,, \tag{153}$$

we can numerically evaluate a MB integral to very high precision using `NIntegrate` as a_0 and find an analytic solution

$$a_0 = \frac{r_1}{r_0} a_1 + \frac{r_2}{r_0} a_2 + \cdots + \frac{r_n}{r_0} a_n\,. \tag{154}$$

Depending on which factors are pulled out before a_0 is calculated, we might need to add γ_E and π to the basis. The larger n is, the more digits we need in a_0 which quickly gets expensive.

In our example, expanding (144) to ϵ^4, we have for the integral part

$$\begin{aligned}\int_{-\frac{1}{2}-\mathrm{i}\infty}^{-\frac{1}{2}+\mathrm{i}\infty} \mathrm{d}z \Big(\cdots\Big) &= 3.2898681336964528729448303332920503784378998024136 \\ &- 2.4041138063191885707994763230228999815299725846810\ \epsilon^1 \\ &+ 2.1646464674222763830320073930823358055495019038375\ \epsilon^2 \\ &- 9.9830729114759243254510727388545706244427677056444\ \epsilon^3 \\ &- 2.8676528331332807519186620879948731784305070422237\ \epsilon^4\end{aligned} \tag{155}$$

calculated to 50 digits accuracy (cf. Listing 14). We can now use PSLQ with the following basis

$$\{\underbrace{\zeta_2}_{w=2},\ \underbrace{\zeta_3}_{w=3},\ \underbrace{\zeta_4}_{w=4},\ \underbrace{\zeta_5,\ \zeta_2\zeta_3}_{w=5},\ \underbrace{\zeta_6,\ \zeta_3\zeta_3}_{w=6}\} \tag{156}$$

to find

$$\int_{-\frac{1}{2}-\mathrm{i}\infty}^{-\frac{1}{2}+\mathrm{i}\infty} \mathrm{d}z \Big(\cdots\Big) = 2\zeta_2 - 2\zeta_3\epsilon^1 + 2\zeta_4\epsilon^2 - \Big(2\zeta_5 - 4\zeta_2\zeta_3\Big)\epsilon^3 - \Big(\frac{17}{2}\zeta_6 + 4\zeta_3^2\Big)\epsilon^4 + \mathcal{O}(\epsilon^5)\,. \tag{157}$$

The basis has the added property that is has constant transcendentality.

In [32], Stefano Laporta calculated 1100 digits of the 4-loop QED correction to $(g-2)_e$ and fitted a basis with more than hundred elements.

Definition 65 (Transcendentality)**.** For each element of the PSLQ basis a we can define a transcendentality $w(a)$. This property is multiplicative, i.e. $w(a_1 \times a_2) = w(a_1) + w(a_2)$. Some transcendentalities can be found in Table 15. Note that even though (156) has what is called constant transcendentality (at each order w is constant), this is not a very useful concept for QCD. However, certain symmetries such as conformal symmetry result in constant transcendentality at the level of the matrix element.

6 Series expansions

The previous example of differential equations was fairly simple, we could guess a $d\log$ form without too much difficulty and our alphabet was $\{1+x, 1-x\}$. While GPLs are fairly well studied objects, many phonologically relevant integrals cannot be expressed in terms of GPLs. Instead they require elliptic integrals (or worse). Even though it is sometimes still possible to evaluate these functions numerically, there is an aspect of diminishing return. What do we gain from an analytic solution if it requires minutes or hours to be evaluated for each phase space point? Instead, we can be pragmatic and prefer a semi-numerical solution in terms of one or more series expansions. We will mostly follow [34] but include further background information and a simple example.

Before we do this, we need to extend our discussion to processes with more than one kinematic variable.

```
integrand=Gamma[1-ep]*Gamma[-ep]*Gamma[-1-ep-z]^2*Gamma[-z]*Gamma[1+z]*
   ↪ Gamma[2+ep+z]/Gamma[-2*ep]/Gamma[-ep-z];

integrandC = integrand/.{
  z -> -1/2 + I * v
};

num = Table[
  NIntegrate[
    SeriesCoefficient[
      integrandC,
      {ep,0,i},
    ],
    {v,-Infinity, Infinity},
    WorkingPrecision -> 100,
    PrecisionGoal -> 50,
    AccuracyGoal -> 50
  ],
  {i, 0, 4}
];
MyPSLQ[num, {
  Zeta[2],
  Zeta[3],
  Zeta[4],
  Zeta[5],Zeta[2]Zeta[3],
  Zeta[6],Zeta[3]Zeta[3]
}]
```

Listing 14: Numerical evaluation and PSLQ

w	#	values
0	1	1
1	1	$\log 2$
2	2	ζ_2, $\log^2 2$
3	3	ζ_3, $\zeta_2 \log 2$, $\log^3 2$
4	5	ζ_4, $\zeta_3 \log 2$, $\zeta_2 \log^2 2$, $\log^4 2$, $\mathrm{Li}_4(\frac{1}{2})$
5	8	ζ_5, $\zeta_2\zeta_3$, $\zeta_4 \log 2$, $\zeta_3 \log^2 2$, $\zeta_2 \log^3 2$, $\log^5 2$, $\log 2\mathrm{Li}_4(\frac{1}{2})$, $\mathrm{Li}_5(\frac{1}{2})$
6	13	ζ_6, $\zeta_3\zeta_3$, $\zeta_5 \log 2$, $\zeta_2\zeta_3 \log 2$, $\zeta_4 \log^2 2$, $\zeta_3 \log^3 2$, $\zeta_2 \log^4 2$, $\log^6 2$, $\zeta_2\mathrm{Li}_4(\frac{1}{2})$, $\log^2 2\mathrm{Li}_4(\frac{1}{2})$, $\log 2\mathrm{Li}_5(\frac{1}{2})$, $\mathrm{Li}_6(\frac{1}{2})$ $H_{0,0,0,0,1,1}(\frac{1}{2})$
w		ζ_i, $\mathrm{Li}_i(\frac{1}{2})$, $\cdots$

Table 15: Some transcendental numbers up to $w = 6$, adapted from [33]

Lemma 66 (Integration paths)**.** Consider a family of integrals that depend on more than one kinematic variable but rather a set $\{s\}$. To transport the boundary condition at some $\{s_0\}$ we define a line segment $\gamma(x)$ that connects our boundary condition to our target. For example,

$$\gamma(x=0) = (\gamma_{s_1}(x), \gamma_{s_2})(x), \cdots) = \{s_0\} \quad \text{and} \quad \gamma(x=1) = \{s\}\,. \tag{158}$$

We can use (54) to write

$$\frac{\partial \vec{I}}{\partial x} = A_x(x,\epsilon)\vec{I} \quad \text{with} \quad A_x = \sum_{s_i\in\{s\}} A_{s_i}(\gamma(x))\frac{\partial \gamma_{s_i}(x)}{\partial x} \tag{159}$$

to reduce our problem to the one-dimensional case.

6.1 Reminder: ordinary differential equations

Theorem 67 (Frobenius method)**.** Consider a p-th order differential equation of the form

$$\mathtt{DE} = \sum_{j=0}^{p} c_j(x)\frac{\partial^j}{\partial x^j} f(x) = 0\,, \tag{160}$$

with $c_p = 1$ w.l.o.g. It is valid for the $c_j(x)$ to have poles as long as they permit a power expansion around $x = 0$. We can obtain a solution of this equation by choosing an ansatz

$$f(x) = x^r \sum_{i=0}^{\infty} a_i x^i \tag{161}$$

and comparing coefficients order-by-order in x. This gives us a linear system of equations for as many a_i as we need which we can solve iteratively.

Proof. We assume that the c_j can be power expanded around $x = 0$ as

$$c_j(x) = x^{r_c}\sum_{k=0}^{\infty} c_{j,k} x^k\,, \tag{162}$$

where r_c is chosen to equal for all c_j. With the derivatives

$$\frac{\partial^j}{\partial x^j} f(x) = x^r \sum_{i=0}^{\infty} a_i x^{i-j}\frac{\Gamma[1+i+r]}{\Gamma[1+i-j+r]} \tag{163}$$

we now have

$$\mathtt{DE} = \sum_{j=0}^{p}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} c_{j,k} a_i \frac{\Gamma[1+i+r]}{\Gamma[1+i-j+r]} x^{i-j+k+r_j+r} = 0\,. \tag{164}$$

We can collect terms by their power in x and get a system of equations for the a_i

$$\mathtt{DE} = \sum_{i=0}^{\infty} x^{i-p+r+r_c} \underbrace{\sum_{k=0}^{p}\sum_{j=0}^{i+k-p} a_j c_{k,i-j+k-p}\frac{\Gamma[1+j+r]}{\Gamma[1+j-k+r]}}_{0}\,. \tag{165}$$

The first non-zero term in this expansion is called the incidentional equation and it will fix r while keeping a_0 free. If $c_{p,0}$ is not zero, this term is at $i = 0$

$$i=0: \qquad a_0 c_{p,0}\frac{\Gamma[1+r]}{\Gamma[1-p+r]} = a_0 c_{p,0}\prod_{n=0}^{p-1}(r-n) = 0\,. \tag{166}$$

It is advisable to choose the largest solution for r to perform recursion and obtain the remaining $p-1$ solutions of the differential equation. Once we have a value for r we can solve the remaining system and obtain as many a_j as we desire. $\square$

Theorem 68 (Reduction of order)**.** After obtaining one solution f_0 for the differential equation, we can obtain another solution f_1 by applying the Frobenius method to

$$\mathtt{DE}' = \sum_{j=0}^{p-1} \underbrace{\sum_{n=0}^{p-1-j} \binom{p-n}{1+j} c_{-n+p}(x) \frac{\partial^{p-n-j-1} f_0(x)}{\partial x^{p-n-j-1}}} \frac{\partial^j}{\partial x^j} f_1'(x) = 0\,, \tag{167}$$

and integrating $f_1 = f_0 \cdot \int \mathrm{d}x f_1'$ (which is trivial since f_1' is a power series). Recursively applying this algorithm will eventually produce all p solutions since $\mathtt{DE}'$ is of lower order than $\mathtt{DE}$.

Proof. Applying the original differential equation to f_1 yields

$$0 = \sum_{j=0}^{p} c_j(x) \frac{\partial^j}{\partial x^j} \left(f_0(x) \cdot \int \mathrm{d}x f_1'(x) \right) \tag{168a}$$

$$= \sum_{j=0}^{p} c_j(x) \binom{j}{n} \frac{\partial^{j-n} f_0}{\partial x^{j-n}} \frac{\partial^n}{\partial x^n} \left(\int \mathrm{d}x f_1'(x) \right) \tag{168b}$$

$$= \sum_{j=0}^{p} c_j(x) \binom{j}{n} \frac{\partial^{j-n} f_0}{\partial x^{j-n}} \frac{\partial^{n-1} f_1'}{\partial x^{n-1}}\,. \tag{168c}$$

Now we only need to re-arrange the summation to arrive at (167)

□

Lemma 69. We can translate a $p \times p$ system of first-order differential equations

$$\frac{\partial \vec{f}}{\partial x} = M \vec{f} \tag{169}$$

into a p-th order differential equation for the first function f_1. Solving this for f_1 will then allow us to obtain the rest of $\vec{f}$. We can gather these into a $n \times p$ matrix F s.t.

$$\frac{\partial F}{\partial x} = MF\,. \tag{170}$$

Proof. By taking derivatives of (169) it is obvious that even higher derivatives can be related back to $\vec{f}$ with different matrices $M^{(j)}$

$$\begin{aligned} \frac{\partial^j \vec{f}}{\partial x^j} &= M^{(j)} \vec{f} \\ &= \frac{\partial}{\partial x} \frac{\partial^{j-1} \vec{f}}{\partial x^{j-1}} = \frac{\partial}{\partial x} \left(M^{(j-1)} \vec{f} \right) = \frac{\partial M^{(j-1)}}{\partial x} \vec{f} + M^{(j-1)} \frac{\partial \vec{f}}{\partial x} = \underbrace{\left(\frac{\partial M^{(j-1)}}{\partial x} + M^{(j-1)} M \right)}_{M^{(j)}} \vec{f}\,. \end{aligned} \tag{171}$$

The top rows $M_{1i}^{(j)}$ of the new matrices define yet another matrix. Depending on how many derivatives we include in this we can obtain two different version: $\tilde{M}$, a $(p \times p)$-matrix, and $\bar{M}$, a $((p+1) \times p)$-matrix.

Assuming it is invertible, $\tilde{M}$ can be used to reconstruct all integrals from the first one as

$$\vec{f} = \tilde{M}^{-1} \left(f_1, \frac{\partial f_1}{\partial x}, \frac{\partial^2 f_1}{\partial^2 x}, \cdots, \frac{\partial^{p-1} f_1}{\partial^{p-1} x} \right)\,. \tag{172}$$

To actually obtain f_1, we want a p-th order differential equation for f_1. Note that since $\tilde{M}$ is assumed to be invertible, $\bar{M}$ has a single element $\vec{c}$ in its kernel, i.e. $\vec{c}\bar{M} = \vec{0}$. Right-multiplying with $\vec{f}$ allows us to write

$$0 = \sum_{j=0}^{p} c_j \tilde{M}_{j,i} \vec{f} = \sum_{j=0}^{p} c_j \frac{\partial^j f_1}{\partial x^j} \tag{173}$$

We can write (172) using the Wronskian matrix W

$$W = \begin{pmatrix} f_1 & \cdots & f_p \\ \partial_x f_1 & \cdots & \partial_x f_p \\ \vdots & \ddots & \vdots \\ \partial_x^{p-1} f_1 & \cdots & \partial_x^{p-1} f_p \end{pmatrix} , \tag{174}$$

as $F = \tilde{M}^{-1} W$. □

Lemma 70 (Inhomogenous differential equations)**.** To obtain the general solution of an inhomogeneous differential equation $\partial_x \vec{f} - M\vec{f} = \vec{\mathcal{I}}$, we need to add the general solution F of the homogeneous equation $\partial_x F - MF = 0$ to a particular solution of the inhomogeneous one. This is done by

$$f_i = \sum_j \left[F \cdot \left(\int \mathrm{d}x F^{-1} \cdot \frac{1}{p} \Big(\vec{\mathcal{I}}, \cdots \vec{\mathcal{I}} \Big) + \mathrm{diag}\Big(c_1, \cdots, c_p \Big) \right) \right]_{ij} . \tag{175}$$

The c_i are the integration constants that need to be fixed by the p boundary conditions. Since we are only considering series expansions this integrals is very straightforward.

Proof. Let us define the $p \times p$ matrices $B = (\vec{\mathcal{I}}, \cdots, \vec{\mathcal{I}})/p$ and $E = \mathrm{diag}(c_1, \cdots, c_p)$. Consider now the derivative of the bracket $G_{ij} = [\cdots]_{ij}$

$$\partial_x G = \partial_x F \cdot \Big(\int \mathrm{d}x F^{-1} \cdot B + E \Big) + B \,. \tag{176}$$

Since F is a solution of the homogeneous differential equation, i.e. $\partial_x F = MF$, we can see that G is a solution of

$$\partial_x G = M \cdot F \cdot \Big(\int \mathrm{d}x F^{-1} \cdot B + E \Big) + B = M \cdot G + B \,. \tag{177}$$

The factor of $1/p$ in B arises from the sum to ensure that $\sum_j B_{ij} = \mathcal{I}_i$. □

Observation 71 (Getting back to Feynman integrals)**.** Since we can rescale integrals by an arbitrary power in ϵ, we can write w.l.o.g. $M = M_0 + M_1\epsilon + M_2\epsilon^2 + \cdots$. By expanding the differential equation order-by-order in ϵ, we have

$$\frac{\partial \vec{I}_k}{\partial x} - M_0 \vec{I}_k = \underbrace{\sum_{j=0}^{k-1} M_{k-j} \vec{I}_j}_{\vec{\mathcal{I}}_k} \,. \tag{178}$$

When working on the k-th order, we will assume that all previous orders are known so that the inhomogeneity of our differential equation $\vec{\mathcal{I}}_k$ is fully known.

By choosing a clever ordering of the integral, i.e. one that makes M as block-triangular as possible we can simplify the differential equations we need to solve considerably. It is fairly obvious that we should solve all subsectors of any given integral firsts since derivatives never evaluate to integrals in higher sectors.

Definition 72 (Coupled integrals)**.** The matrix M_0 can be interpreted as describing a graph G whose nodes are master integrals and whose edges corresponds to dependencies between integrals. If $(M_0)_{ij} \neq 0$, there is an edge from $i \to j$. We consider those integrals coupled that belong to the same *strongly connected component* of the graph, i.e. those sub-graphs where a path from each node to each other node exists.

Lemma 73 (Integration sequence)**.** An optimal integration sequence is obtained by considering the *condensation* $\tilde{G}$ of the graph G, i.e. replacing each strongly connected sub-graph with a single node. We then sort the nodes topologically to ensure that those integrals with the fewest dependencies are solved first.

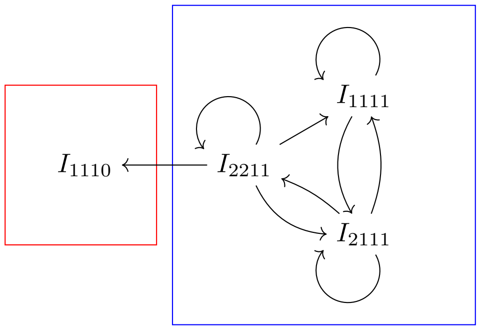
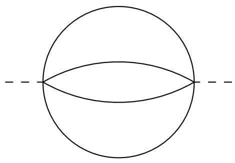


Figure 16: The connection graph corresponding to (180b). When solving this system, we begin with the red integral I_{1110} before simultaneously solving the blue integrals.

Example 74 (Three-loop sunset)**.** Consider the three-loop massive sunset

$$I_{\alpha\beta\gamma\delta} = \; p^2$$



$$= \int \frac{[\mathrm{d}k_1][\mathrm{d}k_2][\mathrm{d}k_3]}{\big[k_1^2-m^2\big]^\alpha \big[k_2^2-m^2\big]^\beta \big[k_3^2-m^2\big]^\gamma \big[(k_1+k_2+k_3+p)^2-m^2\big]^\delta}\,, \tag{179}$$

with $p^2 = s \neq 0$ and $m = 1$. It is fairly easy to find the differential equation in s

$$\frac{\partial}{\partial s}\begin{pmatrix} I_{2211} \\ I_{2111} \\ I_{1111} \\ I_{1110} \end{pmatrix} = \big(M_0 + \epsilon M_1\big)\begin{pmatrix} I_{2211} \\ I_{2111} \\ I_{1111} \\ I_{1110} \end{pmatrix}, \tag{180a}$$

with matrices

$$M_0 = \begin{pmatrix} \frac{18}{(s-16)(s-4)} & -\frac{4(s+20)}{(s-4)(s-16)s} & \frac{36}{(s-4)(s-16)s} & -\frac{2}{(s-16)s} \\ -\frac{3}{s-4} & \frac{s+12}{(s-4)s} & -\frac{6}{(s-4)s} & 0 \\ 0 & -\frac{4}{s} & \frac{2}{s} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{180b}$$

$$M_1 = \begin{pmatrix} \frac{-s^2-16s-64}{(s-16)(s-4)s} & \frac{14(s+20)}{(s-4)(s-16)s} & -\frac{174}{(s-4)(s-16)s} & -\frac{6}{(16-s)s} \\ 0 & \frac{-2s-16}{(s-4)s} & \frac{17}{(s-4)s} & 0 \\ 0 & 0 & -\frac{3}{s} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{180c}$$

The graph corresponding to M_0 is shown in Figure 16

6.2 A full worked example

We can now consider an actual example, start to finish. We have already derived the matrix for the heavy quark bubble $M \equiv M_y$ in (57). While we did guess a canonical form in Example 34 we will pretend to not know this and instead consider the pre-canonical basis.

Example 75 (Boundary condition at $y = 0$)**.** We take our boundary condition at $y = 0$, i.e. $s = 0$. Note that this can be dangerous as the differential equation matrices have a singularity at $y = 0$. However, it turns out that the integrals are well-behaved around this region. Otherwise, we would have to do an asymptotic expansion around $y = 0$.

The first integral, $J_1 \sim I_{001}$, is trivial

$$J_1 = m^{-2+2\epsilon}\epsilon \int [\mathrm{d}k_1]\frac{1}{[(k_1-q)^2-m^2]} = -\epsilon\Gamma(1-\epsilon)\Gamma(-1+\epsilon) \tag{181a}$$

$$= -1 - \epsilon - (1+\zeta_2)\epsilon^2 - (1+\zeta_2)\epsilon^3 - \Big(1+\zeta_2+\frac{7}{4}\zeta_4\Big)\epsilon^4 + \mathcal{O}(\epsilon^5)\,. \tag{181b}$$

The second one, $J_2 \sim I_{011}$, is slightly more complicated. We begin by deriving the $\mathcal{U}$ and $\mathcal{F}$ polynomials as we have learned in the previous section

$$\begin{aligned}
&\mathcal{U} = x_1 + x_2\,, \\
&\mathcal{F} = s\; x_1 x_2 + m^2 \mathcal{U} x_1 + m^2 \mathcal{U} x_2 = s\; x_1 x_2 + m^2 (x_1+x_2)^2 \to m^2(x_1+x_2)^2\,.
\end{aligned} \tag{182}$$

And hence,

$$\begin{aligned}
J_2(y=0) = m^{2\epsilon}\epsilon I_{011} &= -m^{2\epsilon}\epsilon\Gamma(1-\epsilon)\Gamma(2-d/2)\int_0^\infty \mathrm{d}x_1\mathrm{d}x_2\delta(\cdots)\frac{\mathcal{U}^{2-d}}{(\mathcal{F}-\mathrm{i}0^+)^{2-d/2}} \\
&= -\Gamma(1+\epsilon)\Gamma(1-\epsilon)\int_0^\infty \mathrm{d}x_1\mathrm{d}x_2\frac{\delta(1-x_1)}{(x_1+x_2)^2} = -\Gamma(1+\epsilon)\Gamma(1-\epsilon) && \text{(183a)} \\
&= -1 - \zeta_2\epsilon^2 - \frac{7}{4}\zeta_4\epsilon^4 + \mathcal{O}(\epsilon^5)\,. && \text{(183b)}
\end{aligned}$$

Example 76 (Expansion around $y=0$)**.** The differential equation we need to solve is only for $J_2 = J_{2,0} + J_{2,1}\epsilon + J_{2,2}\epsilon^2 + \cdots$

$$\partial_y J_{2,0} = \underbrace{M_{0,21}J_{1,0}}_{\mathcal{I}_0} + M_{0,22}J_{2,0}\,, \tag{184a}$$

$$\partial_y J_{2,k} = \underbrace{M_{1,21}J_{1,k-1} + M_{1,22}J_{2,k-1}}_{\mathcal{I}_k} + M_{0,22}J_{2,k}\,. \tag{184b}$$

The homogeneous solution J_h of $\partial_y J_h = M_{0,22}J_h$ does not depend on the order in ϵ. Following the Frobenius method, we have

$$J_h = \sum_{i=0}^\infty y^{i+r}c_i \tag{185a}$$

$$\partial_x J_h = \sum_{i=-1}^\infty (i+r+1)y^{i+r}c_{i+1} \tag{185b}$$

$$= \frac{1}{2y(y-1)}\sum_{i=0}^\infty y^{i+r}c_i = -\frac{1}{2}\sum_{j=-1}^\infty y^j \sum_{i=0}^\infty y^{i+r}c_i\,. \tag{185c}$$

The first term of this equation at y^{r-1},

$$y^{r-1}\Big(\frac{c_0}{2} + rc_0\Big) = 0\,, \tag{186}$$

requires $r=-1/2$. In general, this would be numerically solved order-by-order. Here, we can write down a closed form solution as

$$c_n = -\frac{(2n)!}{2^{1+2n}(1+n)(n!)^2}c_0\,. \tag{187}$$

Hence, our homogenous solution is

$$J_h = c_0 y^{-1/2} - \frac{1}{2}c_0 y^{1/2} - \frac{1}{8}c_0 y^{3/2} - \frac{1}{16}c_0 y^{5/2} - \frac{5}{128}c_0 y^{7/2} - \frac{7}{256}c_0 y^{9/2} + \mathcal{O}(y^{11/2})\,. \tag{188}$$

Next, we need to derive particular solutions order-by-order.

1. $k=0$: The inhomogeneity is given by $J_{1,0}=-1$

$$\mathcal{I}_0 = \frac{1}{2y(y-1)} \tag{189a}$$

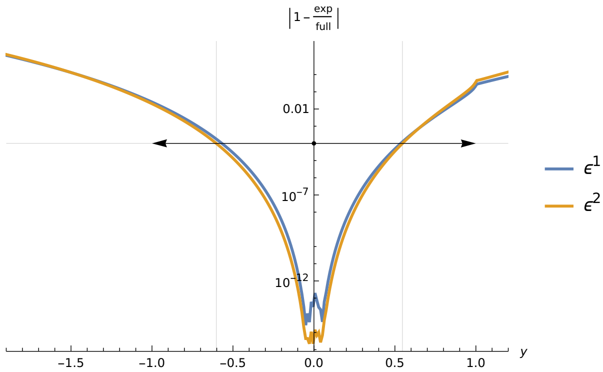


Figure 17: The power-law expansion compared with the correct result for ϵ^1 and ϵ^2. The radius of convergence $R = 1$ is indicated with the arrows.

We can now find

$$J_{p,0} = J_h \int \mathrm{d}y \frac{\mathcal{I}_0}{J_h} = \frac{J_h}{c_0} \int \mathrm{d}y \Bigg(-\frac{y^{-1/2}}{2} - \frac{3y^{1/2}}{4} - \frac{15y^{3/2}}{16} - \frac{35y^{5/2}}{32} - \frac{315y^{7/2}}{256} + \mathcal{O}(y^{9/2}) \Bigg) . \tag{189b}$$

Since $\int y^n = y^{1+n}/(1+n)$,

$$J_{p,0} = -1 . \tag{189c}$$

Our boundary condition requires $J_{2,0}(y=0) = -1$ which means that $c_0 = 0$ and $J_{2,0} = J_{p,0} = -1$.

2. $k = 1$: We now have

$$\mathcal{I}_1 = -1 - y - y^2 - y^3 - y^4 - y^5 + \mathcal{O}(y^6) \tag{190a}$$

$$J_{p,1} = -\frac{2y}{3} - \frac{4y^2}{15} - \frac{16y^3}{105} - \frac{32y^4}{315} - \frac{256y^5}{3465} + \mathcal{O}(y^6) . \tag{190b}$$

Our boundary condition requires $J_{2,1}(y=0) = 0$ which means that $c_0 = 0$ and $J_{2,1} = J_{p,1}$.

3. $k = 2$:

$$\mathcal{I}_2 = -\frac{\zeta_2}{2y} - \frac{\zeta_2}{2} + y\Big(-\frac{\zeta_2}{2} - \frac{2}{3} \Big) + y^2\Big(-\frac{\zeta_2}{2} - \frac{14}{15} \Big) + y^3\Big(-\frac{\zeta_2}{2} - \frac{38}{35} \Big) + \mathcal{O}(y^4) , \tag{191a}$$

$$J_{p,2} = -\zeta_2 - \frac{4y^2}{15} - \frac{8y^3}{35} - \frac{176y^4}{945} - \frac{320y^5}{2079} + \mathcal{O}(y^6) . \tag{191b}$$

We can continue this process until we our preferred order in ϵ.

We can plot the resulting function (expanded up to $\mathcal{O}(y^{15})$ for better convergence) with the exact result we have obtained in Example 34. The resulting ratios are shown in Figure 17. Note that the expansion becomes very bad around $y = \pm 1$. This is due its finite radius of convergence $R = 1$, as we will see next.

Theorem 77 (Fuchs's theorem (applied to Feynman integrals))**.** When expanded around $x = 0$, the general solution of the homogenous equation $\partial_x \vec{f} - M\vec{f} = 0$ has a radius of convergence

$$R = \min \Big\{ |x_i| \ : \ \text{where } x_i \text{ are the poles of } M \Big\} \tag{192}$$

as these specify the radii of convergence of a series expansion of M.

Lemma 78 (Matching of power series)**.** Consider a function $f(x)$ with a power series with known coefficients a_i around x_0 and a radius of convergence R_0

$$f(x) = f_0(x) = (x-x_0)^{r_0} \sum_{i=0}^{\infty} a_i (x-x_0)^i \,. \tag{193}$$

Consider further an expansion around x_1 with yet unknown coefficients b_i and a radius R_1

$$f(x) = f_1(x) = (x-x_1)^{r_1} \sum_{i=0}^{\infty} b_i (x-x_1)^i \,. \tag{194}$$

If there is a point x such that $|x-x_0| < R_0$ and $|x-x_1| < R_1$, we can evaluate $f_0(x)$ and use this as the boundary condition for f_1 to determine the b_i. We now have an expression for f that covers a larger area

$$\begin{aligned} f_{1,2} : \{x \;:\; |x-x_0| < R_0\} \cup \{x \;:\; |x-x_1| < R_1\} &\to \mathbb{C} \\ x &= \begin{cases} f_0(x) & |x-x_0| < |x-x_1| \\ f_1(x) & \text{otherwise} \end{cases} . \end{aligned} \tag{195}$$

Example 79 (Expanding around $y = -1/2$)**.** By setting $y = -1/2 - y'/2$, we can expand around $y = -1/2$ ($y' = 0$) to extend our region of good convergence. This changes the differential equation

$$M' = -\frac{1}{3+4y'+y'^2} \begin{pmatrix} 0 & 0 \\ -1+\epsilon & 1+\epsilon(1+y') \end{pmatrix} , \tag{196}$$

the homogeneous solution

$$J_{h'} = c'_0 - \frac{c'_0}{3} y' + \frac{5c'_0}{18} y'^2 - \frac{13c'_0}{54} y'^3 + \frac{139c'_0}{648} y'^4 - \frac{379c'_0}{1944} y'^5 + \mathcal{O}(y'^6) \,, \tag{197}$$

and the particular solutions

$$J_{p',0} = 1 \,, \tag{198}$$

$$J_{p',1} = \frac{y'}{3} - \frac{y'^2}{9} + \frac{2y'^3}{27} - \frac{5y'^4}{81} + \frac{67y'^5}{1215} + \mathcal{O}(y'^6) \,. \tag{199}$$

The general solution $J_{h'} + J_{p',1}$ has one free parameter, c'_0 that can be fixed through the matching at eg. $y = -1/6$ ($y' = -2/3$)

$$J_{2,1}\Big(y = -\frac{1}{6}\Big) = \frac{255388946}{2447679465} = 0.104339 \,, \tag{200}$$

$$\Rightarrow c'_0 = \frac{185606208569944}{660431033911495} = 0.281038 \,. \tag{201}$$

The new result is shown in Figure 18. Note that M' has poles at $y' = \{-1,-3\}$ ($y = \{1,0\}$) indicating a radius of convergence of $R' = 1$.

Example 80 (Expanding around $y = -1$)**.** We can repeat this one last time and expand around $y = -1 - y'' = -1$ to cover the whole range $-2 < y < 1$. We perform the matching $y = -5/6$ with the result is shown in Figure 19.

Observation 81. Note that we could have jumped from $y = 0$ to $y = -1$ directly without expanding around $y = -1/2$ first. However, as shown in Figure 19, the resulting precision is significantly reduced as the $y = 0$ expansion is less precise at the matching point $y = -1/2$. It is therefore advisible to use smaller steps or techniques that improve convergence such as Möbius transforms (ensure that the nearest pair of singularities are an equal distance from the origin) or Padé approximants (rational functions of fixed degree that can be derived from the Taylor expansion and that usually perform better) [34].

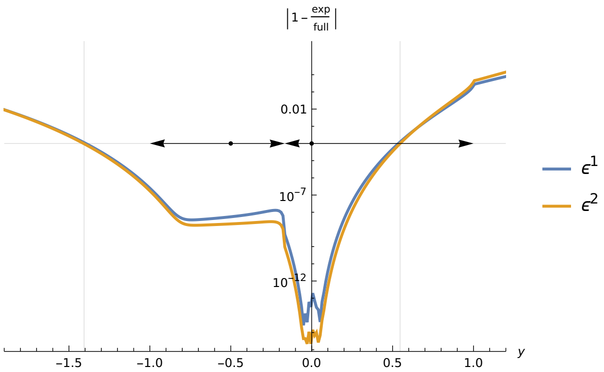
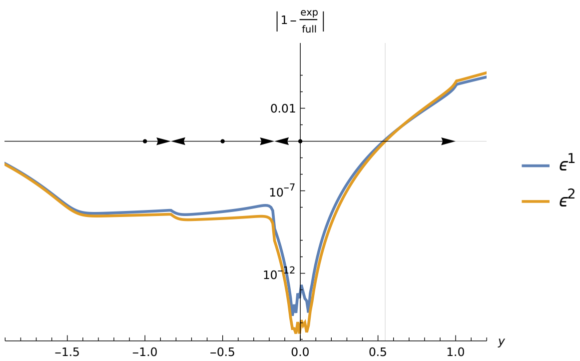
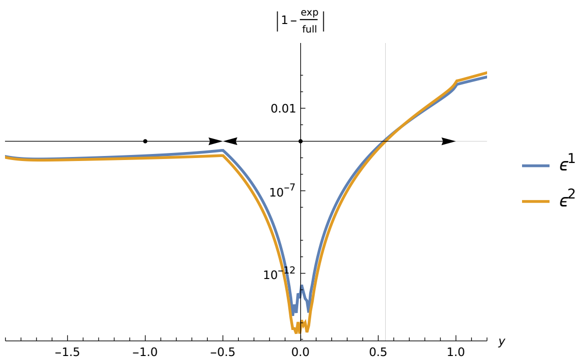


Figure 18: The stiched result expanded around $y = 0$ and $y = -1/2$ (indicated by the dots) and the area in which each result is to be used (indicated by the arrows).



(a) $y = -1$, $y = -1/2$, $y = 0$



(b) $y = -1$, $y = 0$

Figure 19: The stiched result to cover the full domain $y = [-2, 0]$, expanded with three nodes (left panel) or two nodes (right panel).

6.3 Auxiliary mass flow

As we have seen it is possible to solve arbitrarily complicated Feynman integrals using series expansions once we have the differential equation matrix and the boundary conditions. Laporta's algorithm allows us to, at least in principle, find the differential equation matrix for any set of integrals. However, the boundary condition still need to be computed manually. While we could use a tool like `pySecDec` [35, 36, 26, 37] to evaluate these numerical using sector decomposition, this can be numerically challenging. Instead, we can use the method of region in a particularly clever way [38, 39].

Observation 82 (Expansion in large masses)**.** We saw in Example 35 that expanding in a heavy mass results in two regions. In the hard region h all other masses and kinematic invariants are set to zero. In the soft region s the massive propagator is replace with $1/M^2$. Either way, the resulting integral becomes a lot simpler.

Theorem 83 (Auxilary mass)**.** Consider an integral where a large *auxiliary mass* η is added to each propagator, i.e.

$$I = \int \frac{[\mathrm{d}k_1]\cdots[\mathrm{d}k_\ell]}{\mathcal{D}_1\mathcal{D}_2\cdots\mathcal{D}_p} \to I_\eta = \int \frac{[\mathrm{d}k_1]\cdots[\mathrm{d}k_\ell]}{(\mathcal{D}_1-\eta)(\mathcal{D}_2-\eta)\cdots(\mathcal{D}_p-\eta)}\,, \tag{202}$$

with $\mathcal{D}_i = (k_i+p_i)^2 - m_i^2$. For $\eta \gg m_i^2, s_{ij}$ this integral can be considered known and used as a boundary condition.

Proof. Applying the method of regions means that we either have all loop momenta hard, i.e. h^ℓ, or at least one loop momentum soft, i.e. $sX^{\ell-1}$.

- For h^ℓ the propagators become $\mathcal{D}_i = k_i^2 - \eta$, i.e. single-scale vacuum integrals. These are known up to $\ell = 5$ and it is in fact possible to derive them iteratively [40].
- For $sX^{\ell-1}$ we assume w.l.o.g. that k_1 is the soft momentum. This means all propagators of the form $\mathcal{D}_i = (k_1+p_i)^2 - m_i^2 - \eta$ just become $\mathcal{D}_i \to \eta$. In propagators that have other loop momenta, we can safely neglect k_1. This means our k_1 integration is trivial and we are left with an $(\ell-1)$-loop integral.

By recursively applying this algorithm we can solve any I_η. □

Theorem 84 (Auxilary mass flow)**.** We can use IBP reduction to derive a differential equation for the system of $\vec{I}_\eta$

$$\frac{\partial}{\partial\eta}\vec{I}_\eta = M_\eta \vec{I}_\eta\,, \tag{203}$$

and use the boundary condition $\eta \to -\mathrm{i}\infty$ to obtain our integrals at $\eta = \mathrm{i}0^-$ through multiple series expansions at

- $\eta \to -\mathrm{i}\infty$: We fix our boundary conditions at $\eta \to \mathrm{i}\infty$ and perform a power-log expansion

$$\vec{I}_\eta^{\,(\infty)} = \eta^{\vec{a}+\vec{b}\epsilon} \sum_{i=0}^\infty \vec{c}_i(\epsilon)\eta^{-i}\,. \tag{204}$$

  The start of the expansion $\vec{a}$, order of the logarithms $\vec{b}$, and the first term $\vec{c}_0$ are wholly determined by the boundary condition while the remaining $\vec{c}_i(\epsilon)$ are fixed by the differential equation.
- $\eta = \eta_0$: We match and expand a Taylor series around η_0 which is chosen such that $|\eta_0| > |\eta_s|$ for all singularities of the differential equation matrix $\eta_s \neq 0$.
- $\eta = \eta_i$: We match and expand Taylor series with overlapping radii of convergence.
- $\eta = \eta_N$: The last value we expand around is chosen such that $|\eta_N| < |\eta_s|$. Here the expansion takes the form

$$\vec{I}_\eta^{\,(N)} = \underbrace{\sum_{i=0}^\infty \vec{c}_i(\epsilon)\eta^i}_{\text{homogeneous}} + \underbrace{\eta^{\vec{a}+\vec{b}\epsilon}\sum_{i=0}^\infty \vec{d}_i(\epsilon)\eta^i}_{\text{subtopologies}}\,. \tag{205}$$

- $\eta = \mathrm{i}0^-$: To go from our last expansion to zero, we have

$$\vec{I} = \vec{c}_0(\epsilon)\,, \tag{206}$$

since in dimensional regularisation

$$\lim_{\eta\to\mathrm{i}0^-} \eta^{a+b\epsilon} = 0 \quad \text{for} \quad b \neq 0\,. \tag{207}$$

Observation 85. Note that we are expanding around $\eta = -\mathrm{i}\infty$ rather than just $\eta = \infty$. Singularities of the original integral lie on the real axis in η and therefore $\vec{I}_\eta$ is finite as long as $\Im\eta \neq 0$. Hence, choosing $\eta = -\mathrm{i}\infty$ may complicate matters as we have to work with complex masses, it greatly simplifies the running $\eta \to \mathrm{i}0^-$.

Lemma 86 (Step size)**.** We already know that the first matching point, η_0, needs to be bigger than the largest singularity while the last, η_N needs to be smaller than the smallest non-zero singularity. In practice, we choose a factor R and then define

$$\eta_0 = -\mathrm{i}R\max\{|\eta_s| \;:\; \eta_s \neq \infty \text{ singularities of } M_\eta\}\,, \tag{208a}$$

$$\eta_N = -\frac{\mathrm{i}}{R}\min\{|\eta_s| \;:\; \eta_s \neq 0 \text{ singularities of } M_\eta\}\,. \tag{208b}$$

Next, we choose the intermediary values η_i such that

$$\frac{\eta_{i+1}}{\eta_i} = \frac{R-1}{R} \quad \text{for} \quad 0 \le i < N\,. \tag{208c}$$

These equations also define the number N of steps we are taking.

Example 87 (Massless bubble)**.** Consider the integral family

$$I_{\alpha\beta} = \int \frac{[\mathrm{d}k]}{\big[k^2\big]^\alpha \big[(k-p)^2\big]^\beta}\,, \tag{209}$$

with $p^2 = s \equiv 1$. This family has a single master integral I_{11}. Adding the auxiliary mass modifies this to

$$I_{\eta,\alpha\beta} = \int \frac{[\mathrm{d}k]}{\big[k^2-\eta\big]^\alpha \big[(k-p)^2-\eta\big]^\beta}\,, \tag{210}$$

which now has two master integrals $I_{\eta,10}$ and $I_{\eta,11}$. The differential equation matrix for $\vec{I}_\eta$ is, as in (51)

$$M_\eta = \begin{pmatrix} \frac{1-\epsilon}{\eta} & 0 \\ -\frac{2-2\epsilon}{\eta(4\eta-1)} & \frac{2(1-2\epsilon)}{4\eta-1} \end{pmatrix}\,. \tag{211}$$

We can hence expect singularities at $\eta = 0$, $\eta = 1/4$, $\eta = \infty$

The first boundary condition is just a tadpole

$$I_{\eta,10} = \int \frac{[\mathrm{d}k]}{k^2-\eta} = -\eta^{1-\epsilon}\Gamma(1-\epsilon)\Gamma(-1+\epsilon)\,. \tag{212}$$

For the second, we expand in $\eta \sim \infty$. The soft region s vanishes as both propagators just become $1/\eta$ while the hard region h is

$$I_{\eta,11} \sim \int \frac{[\mathrm{d}k]}{\big[k^2-\eta\big]^2} = \eta^{-\epsilon}\Gamma(1-\epsilon)\Gamma(\epsilon) + \mathcal{O}\Big(\frac{1}{\eta}\Big)\,. \tag{213}$$

Next, we need to choose our expansion points. The poles of M_η are at $\eta = 0$ and $\eta = 1/4$. Using $R = 2$ in (208) we have $\eta_0 = -\mathrm{i}/2$, $\eta_1 = -\mathrm{i}/4$, and $\eta_2 = \eta_N = -\mathrm{i}/2$.

- $\eta \to -\mathrm{i}\infty$: We have $\vec{a} = (1,0)$ and $\vec{b} = (-1,-1)$. We can now solve for $\vec{c}$ and find

$$\vec{c}_0 = \begin{pmatrix} -\Gamma(1-\epsilon)\Gamma(-1+\epsilon) \\ \Gamma(1-\epsilon)\Gamma(\epsilon) \end{pmatrix}, \tag{214a}$$

$$\vec{c}_1 = \Big(0, \frac{1}{6} + \mathcal{O}(\epsilon^2)\Big)^T, \tag{214b}$$

$$\vec{c}_2 = \Big(0, \frac{1}{60} + \frac{1}{60}\epsilon + \mathcal{O}(\epsilon^2)\Big)^T, \tag{214c}$$

$$\vec{c}_3 = \Big(0, \frac{1}{420} + \frac{1}{280}\epsilon + \mathcal{O}(\epsilon^2)\Big)^T. \tag{214d}$$

Evaluating this for the next point $\eta = -\mathrm{i}/2$ gives for the second integral

$$I^{(\infty)}_{\eta,11}(\eta = -\mathrm{i}/2) = \frac{1}{\epsilon} + (0.632066 + 1.88709\mathrm{i}) + (0.0555556 + 1.18759\mathrm{i})\epsilon + \mathcal{O}(\epsilon^2). \tag{215}$$

- $\eta = -\mathrm{i}/2$: We perform a normal Taylor expansion with the $(\eta + \mathrm{i}/2)^0$ term given by (215)

$$\begin{aligned} I^{(0)}_{\eta,11} = &\Big(\frac{1}{\epsilon} + (0.632066 + 1.88709\mathrm{i}) + (0.0555556 + 1.18759\mathrm{i})\epsilon\Big) \\ &+ (\eta + \mathrm{i}/2)\Big((0.5714 - 1.77538\mathrm{i}) + (3.68432 - 0.0178779\mathrm{i})\epsilon\Big) \\ &+ (\eta + \mathrm{i}/2)^2\Big(-(1.39587 + 0.983636\mathrm{i}) - (0.388233 + 4.17847\mathrm{i})\epsilon\Big) \\ &+ \mathcal{O}\Big((\eta + \mathrm{i}/2)^3, \epsilon^2\Big) \end{aligned} \tag{216}$$

Evaluating this at $\eta = -\mathrm{i}/4$ we have

$$I^{(0)}_{\eta,11}(-\mathrm{i}/4) = \frac{1}{\epsilon} + (1.18638 + 2.13174\mathrm{i}) + (0.0584251 + 2.50302\mathrm{i})\epsilon + \mathcal{O}(\epsilon^2). \tag{217}$$

- $\eta = -\mathrm{i}/4$: We perform a normal Taylor expansion with the $(\eta + \mathrm{i}/4)^0$ term given by (217)

$$\begin{aligned} I^{(1)}_{\eta,11} = &\Big(\frac{1}{\epsilon} + (1.18638 + 2.13174\mathrm{i}) + (0.0584251 + 2.50302\mathrm{i})\epsilon\Big) \\ &+ (\eta + \mathrm{i}/4)\Big((1.63896 - 2.76086\mathrm{i}) + (7.62454 + 0.251576\mathrm{i})\epsilon\Big) \\ &+ (\eta + \mathrm{i}/4)^2\Big(-(2.56095 + 4.19991\mathrm{i}) + (3.18517 - 14.0005\mathrm{i})\epsilon\Big) \\ &+ \mathcal{O}\Big((\eta + \mathrm{i}/4)^3, \epsilon^2\Big) \end{aligned} \tag{218}$$

Evaluating this at $\eta = -\mathrm{i}/8$ we have

$$I^{(1)}_{\eta,11}(-\mathrm{i}/8) = \frac{1}{\epsilon} + (1.57145 + 2.42639\mathrm{i}) - (0.0708639 - 3.73778\mathrm{i})\epsilon + \mathcal{O}(\epsilon^2). \tag{219}$$

- $\eta = \mathrm{i}0^-$: We begin with a power expansion for the homogeneous equation

$$\frac{\partial I^{(N)}_{\eta,11,h}}{\partial \eta} = \frac{2(1-2\epsilon)}{4\eta - 1} I^{(N),h}_{\eta,11}, \tag{220}$$

and find

$$\begin{aligned} I^{(N)}_{\eta,11,h} = &\Big(\frac{c_{2,0,-1}}{\epsilon} + c_{2,0,0} + \epsilon c_{2,0,1} + \mathcal{O}(\epsilon^2)\Big) \\ &+ \eta\Big(-\frac{2c_{2,0,-1}}{\epsilon} + 4c_{1,0,-1} - 2c_{2,0,0} + \epsilon(4c_{2,0,0} - 2c_{2,0,1} + \mathcal{O}(\epsilon^2))\Big) \\ &+ \eta^2\Big(-\frac{2c_{2,0,-1}}{\epsilon} - 2c_{2,0,0} + \epsilon(8c_{2,0,-1} - 2c_{2,0,1} + \mathcal{O}(\epsilon^2))\Big) + \mathcal{O}(\eta^3). \end{aligned} \tag{221}$$

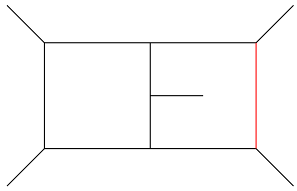


Figure 20: The two-loop pentabox of [39] with all internal and external masses vanishing. The red line is a good candidate for the auxiliary mass.

Next, build the full solution with $a_2 = 1$ and $b_2 = -1$, matched at $\eta = -\mathrm{i}/8$ with (219) which fixes the remaining free coefficients

$$\begin{aligned} I^{(N)}_{\eta,11} =& \Big(\frac{1}{\epsilon} + (2 + 3.14159\mathrm{i}) - (0.934802 - 6.28319\mathrm{i})\epsilon\Big) \\ &+ \eta\Big(-\frac{2}{\epsilon} - (0 + 6.28319\mathrm{i}) + 9.8696\epsilon\Big) \\ &+ \eta^2\Big(-\frac{2}{\epsilon} - (4 + 6.28319\mathrm{i}) + (9.8696 - 12.5664\mathrm{i})\epsilon\Big) \\ &+ \eta^{1-\epsilon}\Big(\frac{2}{\epsilon} + 2. + 5.28987\epsilon\Big) + \eta^{2-\epsilon}\Big(\frac{2}{\epsilon} + 3. + 6.78987\epsilon\Big) + \mathcal{O}\Big(\eta^3, \epsilon^2\Big)\,. \end{aligned} \tag{222}$$

We can now read of the final answer for I_{11}

$$I_{11} = \frac{1}{\epsilon} + (2 + 3.14159\mathrm{i}) - (0.934802 - 6.28319\mathrm{i})\epsilon + \mathcal{O}(\epsilon^2)\,. \tag{223}$$

Observation 88. Note how we started with one master integral I_{11} and had to actually calculate two, $I_{\eta,10}$ and $I_{\eta,11}$. In this example, the integrals were simple enough so that this was not a problem but in a real-life example this might not be the case. Consider for example the diagram in Figure 20 [39]. The original topology contained 108 master integrals. Adding an η to every propagator increases this to 476 integrals. Instead we could add η only to some propagator(s). If we chose the highlighted line, we only have 176 masters which greatly speeds up the computation. Of course, in cases where we have an internal massive line such as an electoweak boson or top quark, we should use its mass as the auxiliary mass.

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