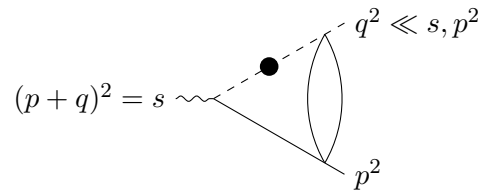


# Practical guide to analytic loop integration

## Exercise Sheet 3

### Exercise 1: top quark decay

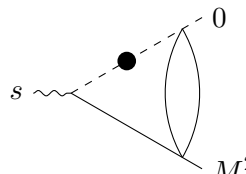
Consider the following integral that appears in the two-loop calculation of the top decay



$$(1)$$

The dashed line corresponds to light  $b$  quark and the solid line to the heavy  $t$  quark.

- Find the momentum regions that contribute to this process.
- Show that the hard region does in fact equal the same calculation with massless  $b$  quarks, i.e.  $m = 0$ .
- The result for  $m = 0$  can be found in the literature as



$$(2)$$

$$= \frac{(M^2)^{-2\epsilon}}{M^2 - s} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \frac{1+y}{y} H_1(y) + 4 \right. \\ \left. + \frac{2(1+y)}{y} \left( H_1(y) + H_{1,1}(y) \right) + \frac{2}{y} \frac{1+y}{1-y} \left( H_{0,1}(y) - y\zeta_2 \right) \right].$$

Show by explicit calculation of the remaining regions that the complete integral has no  $1/\epsilon^2$  pole but a  $\log(m^2/M^2)$ .

**SOLUTION:** We write the integral as

$$I = \int [dk_1][dk_2] \frac{1}{[k_1^2 + 2k_1 \cdot p][k_1^2 - 2k_1 \cdot q]^2 [k_2^2 - M^2][(k_1 - k_2)^2 - M^2]}$$

with  $p^2 = M^2$  and  $q^2 = m^2 \sim \lambda^2$ .

- We have two loop momenta that can scale differently. Hence, a region is fully specified by six numbers

$$k_1 \sim (\lambda^a, \lambda^b, \lambda^c) \quad \text{and} \quad k_2 \sim (\lambda^d, \lambda^e, \lambda^f).$$

We obviously have a hard region  $k_1 \sim k_2 \sim (1, 1, 1)$ . We further have a collinear-hard region  $k_1 \sim (\lambda^2, 1, \lambda)$  and  $k_2 \sim (1, 1, 1)$ . All other regions are scaleless. For example the collinear-collinear region  $k_1 \sim k_2 \sim (\lambda^2, 1, \lambda)$

$$I_{cc} = \frac{\lambda^4}{M^4} \int [dk_1][dk_2] \frac{1}{[2k_1 \cdot p_+][k_1^2 - 2k_1 \cdot q]} = 0$$

b) We have with  $q = q_- + q_\perp$  with  $q_- \sim 1$  and  $q_\perp \sim \lambda$

$$\begin{aligned} I_{hh} &= \int [dk_1][dk_2] \frac{1}{[k_1^2 + 2k_1 \cdot p][k_1^2 - 2k_1 \cdot (q_- + \lambda q_\perp)]^2 [k_2^2 - M^2][(k_1 - k_2)^2 - M^2]} \\ &= \int [dk_1][dk_2] \frac{1}{[k_1^2 + 2k_1 \cdot p][k_1^2 - 2k_1 \cdot q_-]^2 [k_2^2 - M^2][(k_1 - k_2)^2 - M^2]} + \mathcal{O}(\lambda). \end{aligned}$$

Since  $q_-^2 = 0$ , we now have the massless result evaluated at  $q_-$  instead of  $q$ .

c) The other region,  $I_{ch}$ , is

$$I_{ch} = \int [dk_1][dk_2] \frac{1}{[2k_{1-} \cdot p_+][k_1^2 - 2(q_- \cdot k_{1+} + q_\perp \cdot k_{1\perp})]^2 [k_2^2 - M^2][k_2^2 - 2k_{1-} \cdot k_{2+} - M^2]} + \mathcal{O}(\lambda).$$

In the scalar products between  $q$  and  $k_1$  we can just take the full  $k_1$  because the  $q$  will project out the necessary component. The  $k_{1-} \cdot k_{2+}$  is, however, a problem.

$$I_{ch} = \int [dk_1][dk_2] \frac{1}{[2k_1 \cdot p_+][k_1^2 - 2k_1 \cdot (q_- + q_\perp)]^2 [k_2^2 - M^2][k_2^2 - 2k_{1-} \cdot k_{2+} - M^2]} + \mathcal{O}(\lambda).$$

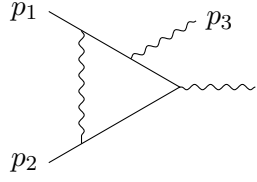
By solving first the  $k_2$  integration, we write  $k_{1-} \cdot k_{2+} = k_{1-} \cdot k_2$

$$\begin{aligned} I_{ch} &= (M^2)^{-\epsilon} \Gamma(1 - \epsilon) \Gamma(\epsilon) \int dx_1 dx_2 \frac{\delta(\dots)}{(x_1 + x_2)^2} \int [dk_1] \frac{1}{[2k_1 \cdot p_+][k_1^2 - 2k_1 \cdot (q_- + q_\perp)]^2} + \mathcal{O}(\lambda) \\ &= 2^\epsilon (M^2)^{-\epsilon} \Gamma(1 - \epsilon)^2 \Gamma(\epsilon) \Gamma(1 + \epsilon) \int dx_1 dx_2 dy_1 dy_2 \frac{\delta(\dots) \delta(\dots)}{(x_1 + x_2)^2} \\ &\quad y_1^{-1+\epsilon} (2m^2 y_1 + (m^2 + M^2 - s) y_2)^{-1-\epsilon} \\ &= -(M^2)^{-2\epsilon} \left( \frac{m^2}{M^2} \right)^{-\epsilon} \frac{\Gamma(1 - \epsilon)^2 \Gamma(\epsilon)^2}{m^2 + M^2 - s} = -(M^2)^{-2\epsilon} \left( \frac{m^2}{M^2} \right)^{-\epsilon} \frac{\Gamma(1 - \epsilon)^2 \Gamma(\epsilon)^2}{M^2 - s} + \mathcal{O}(\lambda) \\ &= \frac{(M^2)^{2\epsilon}}{M^2 - s} \left[ -\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log \frac{m^2}{M^2} + \mathcal{O}(\epsilon^0) \right] + \mathcal{O}(\lambda). \end{aligned}$$

The  $1/\epsilon^2$  pole cancels exactly the one in (2)

**Exercise 2:** Soft approximation

Consider the following diagram



$$Q = \int [dk] \frac{1}{[k^2] [(k-p_1)^2 - m^2] [(k-p_1+p_3)^2 - m^2] [(k+p_2)^2 - m^2]} \quad (3)$$

with the energy of the photon soft compared to  $s = Q^2$  and  $p_1^2 = p_2^2 = m^2 \ll s$ . This can be phrased in a Lorentz invariant way by requiring that all invariants

$$\sigma_{i3} = 2p_i \cdot p_3 \sim m \sim \lambda. \quad (4)$$

- Use the method of regions in the parametric representation to find all six regions that contribute to this integral.
- Consider the region  $\vec{r}^{(1)} = (0, -1, -1, 1)$ , i.e.  $\mathcal{P}_1 \sim 1$ ,  $\mathcal{P}_2 \sim \lambda^{-1}$ ,  $\mathcal{P}_3 \sim \lambda^{-1}$ ,  $\mathcal{P}_4 \sim \lambda$ . Show that this integral is not finite in dimensional regularisation.

Such behaviour is not uncommon and usually points to a broken symmetry eg. in SCET. It is usually addressed by using analytic regularisation, i.e.

$$\begin{aligned} & \int [dk] \frac{1}{[k^2] [(k-p_1)^2 - m^2] [(k-p_1+p_3)^2 - m^2] [(k+p_2)^2 - m^2]} \\ \rightarrow & (-\nu^2)^\eta \int [dk] \frac{1}{[k^2] [(k-p_1)^2 - m^2] [(k-p_1+p_3)^2 - m^2]^{1+\eta} [(k+p_2)^2 - m^2]}. \end{aligned} \quad (5)$$

We have introduced an additional regulator  $\eta$  that we will take to zero as soon as we have added all regions. Crucially,  $\eta \rightarrow 0$  needs to be done *before*  $\epsilon \rightarrow 0$ .

- Calculate the integral, up to  $\mathcal{O}(\lambda^0)$ . Add all regions and set  $\eta \rightarrow 0$  and finally  $\epsilon \rightarrow 0$ .

**SOLUTION:** We have

$$I = \Gamma(1-\epsilon)\Gamma(2+\epsilon) \int dx_1 dx_2 dx_3 dx_4 \delta(\dots) (x_1 + x_2 + x_3 + x_4)^{2\epsilon} \left[ -s(x_2 + x_3)x_4 + m^2(x_2 + x_3 + x_4)^2 + \sigma_{13}(x_1x_3 + 2x_4x_3 + x_2x_4) + \sigma_{23}(x_2 + 2x_3)x_4 \right]^{-\epsilon-2}$$

- Using `asy` and `QHull`

```
Get ["/path/to/asy2.1.m"];
SetOptions [QHull, Executable -> "/path/to/qhull"];
props = {
  k1^2,
  (k1 - p1)^2 - m^2,
```

```

(k1 - p1 + p3)^2 - m^2,
(k1 + p2)^2 - m^2
};
onshell = {
  p1^2 -> m^2, p2^2 -> m^2, p3^2 -> 0,
  p1 p2 -> 1/2 (-2 m^2 + s - σ13 - σ23),
  p1 p3 -> σ13/2,
  p2 p3 -> σ23/2,
};

AlphaRepExpand[{k1},
  props,
  onshell,
  {σ13 -> x, σ23 -> x, m -> x, s -> 1}
]
(* Asy2.1 *)
(* Variables for UF: {k1, p1, p2, p3} *)
(* {{0, -1, -1, 1}, {0, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 2},
  ↪ {0, 2, 2, 0}, {0, 3, 3, 1}} *)

```

we find

$$\begin{aligned}
\vec{r}^{(1)} &= (0, -1, -1, 1), \\
\vec{r}^{(2)} &= (0, 0, 0, 0), \\
\vec{r}^{(3)} &= (0, 0, 0, 1), \\
\vec{r}^{(4)} &= (0, 0, 1, 2), \\
\vec{r}^{(5)} &= (0, 2, 2, 0), \\
\vec{r}^{(6)} &= (0, 3, 3, 1).
\end{aligned}$$

b) In  $\vec{r}^{(1)}$ , we have  $x_1 \sim 1$ ,  $x_2 \sim x_3 \sim \lambda^{-1}$ ,  $x_4 \sim m \sim \sigma_{13} \sim \sigma_{23} \sim \lambda$ . Hence up to a global pre-factor of  $\lambda$

$$\begin{aligned}
I_1 &\propto \int dx_1 dx_2 dx_3 dx_4 \delta(\dots) (x_2 + x_3)^{2\epsilon} \left[ -s(x_2 + x_3)x_4 + m^2(x_2 + x_3)^2 + \sigma_{13}x_1x_3 \right]^{-\epsilon-2} \\
&\propto \int dx_2 dx_3 \delta(\dots) x_3^{-1} (x_2 + x_3)^{-1}.
\end{aligned}$$

This integral integrates to  $\Gamma(0)$  which is not finite.

c) We find after setting the power of the third propagator to  $1 + \eta$

$$\begin{aligned}
I_1 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \\
&\quad \int d\vec{x} x_3^\eta (x_2+x_3)^{\eta+2\epsilon} \left[ m^2(x_2+x_3)^2 - s x_4(x_2+x_3) + \sigma_{13} x_1 x_3 \right]^{-\eta-\epsilon-2} \\
I_2 &= (-\nu^2)^\eta (-s)^{-\eta-\epsilon-2} \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \\
&\quad \int d\vec{x} x_3^\eta x_4^{-\eta-\epsilon-2} (x_2+x_3)^{-\eta-\epsilon-2} (x_1+x_2+x_3+x_4)^{\eta+2\epsilon} \\
I_3 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \int d\vec{x} x_3^\eta (x_1+x_2+x_3)^{\eta+2\epsilon} \left[ \sigma_{13} x_1 x_3 - s(x_2+x_3)x_4 \right]^{-\eta-\epsilon} \\
I_4 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \int d\vec{x} x_3^\eta (x_1+x_2)^{\eta+2\epsilon} \left[ m^2 x_2^2 - s x_4 x_2 + \sigma_{13} x_1 x_3 \right]^{-\eta-\epsilon-2} \\
I_5 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \int d\vec{x} x_3^\eta x_4^{-\eta-\epsilon-2} (x_1+x_4)^{\eta+2\epsilon} \left[ -s(x_2+x_3) + m^2 x_4 \right]^{-\eta-\epsilon-2} \\
I_6 &= (-\nu^2)^\eta \frac{\Gamma(1-\epsilon)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+1)} \int d\vec{x} x_3^\eta x_1^{\eta+2\epsilon} \left[ m^2 x_4^2 - s(x_2+x_3)x_4 + \sigma_{13} x_1 x_3 \right]^{-\eta-\epsilon-2}
\end{aligned}$$

All of these integrals can be trivially solved

$$\begin{aligned}
I_1 &= \frac{1}{(-s)\sigma_{13}} \left( \frac{-\nu^2}{m^2} \right)^\eta \left( \frac{\mu^2}{m^2} \right)^\epsilon \frac{\Gamma(\eta)\Gamma(1-\epsilon)\Gamma(\epsilon+\eta)}{\Gamma(\eta+1)^2} \\
I_2 &= \frac{1}{(-s)(-s)} \left( \frac{-\nu^2}{-s} \right)^\eta \left( \frac{\mu^2}{-s} \right)^\epsilon \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(-\epsilon-\eta-1)\Gamma(\epsilon+\eta+2)}{\Gamma(\eta+2)\Gamma(-2\epsilon-\eta)} \\
I_3 &= \frac{1}{(-s)\sigma_{13}} \left( \frac{-\nu^2}{\sigma_{13}} \right)^\eta \left( \frac{\mu^2}{\sigma_{13}} \right)^\epsilon \frac{\Gamma(-\epsilon)^2\Gamma(-\epsilon-\eta)\Gamma(\epsilon+\eta+1)}{\Gamma(\eta+1)\Gamma(-2\epsilon-\eta)} \\
I_4 &= \frac{1}{(-s)\sigma_{13}} \left( \frac{-\nu^2}{\sigma_{13}} \right)^\eta \left( \frac{\mu^2}{m^2} \right)^\epsilon \frac{\Gamma(-\eta)\Gamma(1-\epsilon)\Gamma(-2\epsilon)\Gamma(\epsilon)}{\Gamma(-2\epsilon-\eta)} \\
I_5 &= \frac{1}{(-s)(-s)} \left( \frac{-\nu^2}{-s} \right)^\eta \left( \frac{\mu^2}{m^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)\Gamma(\epsilon)\Gamma(-2\epsilon-\eta-1)}{\Gamma(-2\epsilon-\eta)} \\
I_6 &= \frac{1}{(-s)\sigma_{13}} \left( \frac{-\nu^2}{\sigma_{13}} \right)^\eta \left( \frac{\mu^2}{m^2} \frac{s^2}{\sigma_{13}^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)\Gamma(-2\epsilon)\Gamma(\epsilon)\Gamma(2\epsilon+\eta+1)}{\Gamma(\eta+1)}
\end{aligned}$$

Clearly,  $I_1$ ,  $I_3$ ,  $I_4$ , and  $I_6$  start at  $\lambda^{-1}$  and need to be expanded one order higher. This results in

$$\begin{aligned}
I &= \frac{1}{\lambda} \frac{1}{-s\sigma_{13}} \left( \frac{1}{\epsilon} \log \frac{m^2}{-s} + \log \frac{m^2}{-s} \log \frac{-s\mu^2}{\sigma_{13}^2} - \zeta_2 \right) \\
&\quad - \frac{\sigma_{13} + \sigma_{23}}{\sigma_{13}} \frac{1}{s^2} \left( \frac{1}{\epsilon} \left( \log \frac{m^2}{-s} + 1 \right) + \log \frac{m^2}{-s} \left( \log \frac{-s\mu^2}{\sigma_{13}^2} - 4 \right) - \zeta_2 + 2 \log \frac{m\mu}{\sigma_{13}} - 2 \right)
\end{aligned}$$