Practical guide to analytic loop integration
Exercise Sheet 2

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https://yannickulrich.gitlab.io/loop-integration

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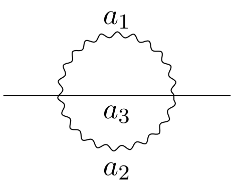
Exercise 1: Sunset diagram Consider the following integral

$$I(a_1, a_2, a_3; \mathcal{N}) = \mathcal{N} \times$$



$$= \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{\mathcal{N}}{\big[k_1^2\big]^{a_1} \big[k_2^2\big]^{a_2} \big[(k_1 - k_2 - p)^2 - m^2\big]^{a_3}} \tag{1}$$

with $p^2 = m^2 \neq 0$ and an arbitrary numerator $\mathcal{N}$.

a) Find a complete family. We know that it will need $\ell(1 + \ell + 2\rho)/2 = 5$ propagators.

b) Identify the sectors in which all integrals vanish.

c) Consider the IBP generated through $\partial_{k_1^\mu}(k_1^\mu I)$. Use it to show that

$$\int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{k_2 \cdot p}{\big[k_1^2\big] \big[k_2^2\big] \big[(k_1 - k_2 - p)^2 - m^2\big]^2} = \frac{3-d}{2} \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2\big] \big[k_2^2\big] \big[(k_1 - k_2 - p)^2 - m^2\big]} \tag{2}$$

d) Now find all six seed identities as a function of $a_1, \ldots, a_5$.

e) Implement Laporta's algorithm to solve the system up to $r = 3$ and $s = 1$ for sector 7 and its subsectors.

SOLUTION:

a) We choose to complete the family as follow

$$I(a_1, a_2, a_3, a_4, a_5) = \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{\big[k_1^2\big]^{a_1} \big[k_2^2\big]^{a_2} \big[(k_1 - k_2 - p)^2 - m^2\big]^{a_3} \big[k_1 \cdot p\big]^{a_4} \big[k_2 \cdot p\big]^{a_5}} \,.$$

Note that this is not a unique solution

b) The sectors 1-6 are zero

$$
\begin{aligned}
I(a_1,0,0,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{1}{\left[k_1^2\right]^{a_1}}\,,\\
I(0,a_2,0,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{1}{\left[k_2^2\right]^{a_2}}\,,\\
I(a_1,a_2,0,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{1}{\left[k_1^2\right]^{a_1}\left[k_2^2\right]^{a_2}}\,,\\
I(0,0,a_3,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{1}{\left[(k_1-k_2-p)^2-m^2\right]^{a_3}}\,,\\
I(a_1,0,a_3,0,0) &= \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{1}{\left[k_1^2\right]^{a_1}\left[(k_1-k_2-p)^2-m^2\right]^{a_3}}\,.
\end{aligned}
$$

c) From

$$
0 = \int [\mathrm{d}k_1][\mathrm{d}k_2]\partial_{k_1^\mu}\left(k_1^\mu \frac{1}{\left[k_1^2\right]^{a_1}\left[k_2^2\right]^{a_2}\left[(k_1-k_2-p)^2-m^2\right]^{a_3}\left[k_1\cdot p\right]^{a_4}\left[k_2\cdot p\right]^{a_5}}\right)
$$

we find

$$
0 = (d-2a_1-a_3-a_4) - a_3\mathbf{1}^-\mathbf{3}^+ + a_3\mathbf{2}^-\mathbf{3}^+ + 2a_3\mathbf{5}^-\mathbf{3}^+\,.
$$

with $a_1=a_2=a_3=1$ and $a_4=a_5=0$, we find our relation

$$
\begin{aligned}
0 &= (d-3)I(1,1,1,0,0) - I(0,1,2,0,0) + I(1,0,2,0,0) + 2I(1,1,2,0,-1)\\
\Rightarrow I(1,1,2,0,-1) &= \frac{3-d}{2}I(1,1,1,0,0)\,.
\end{aligned}
$$

d) We find

$$
\begin{aligned}
0 &= -2a_1 - a_3 - a_4 + d - a_3\mathbf{1}^-\mathbf{3}^+ + a_3\mathbf{2}^-\mathbf{3}^+ + 2a_3\mathbf{5}^-\mathbf{3}^+\\
0 &= -a_1 + a_3 - a_1\mathbf{2}^-\mathbf{1}^+ + a_1\mathbf{3}^-\mathbf{1}^+ + 2a_1\mathbf{4}^-\mathbf{1}^+ - 2a_1\mathbf{5}^-\mathbf{1}^+ - a_3\mathbf{1}^-\mathbf{3}^+\\
&\qquad + a_3\mathbf{2}^-\mathbf{3}^+ + 2a_3\mathbf{4}^-\mathbf{3}^+ - a_4\mathbf{5}^-\mathbf{4}^+\\
0 &= -2a_1\mathbf{4}^-\mathbf{1}^+ - 2a_3\mathbf{4}^-\mathbf{3}^+ + 2a_3\mathbf{5}^-\mathbf{3}^+ + 2a_3m^2\mathbf{3}^+ - a_4m^2\mathbf{4}^+\\
0 &= -a_2 + a_3 - a_2\mathbf{1}^-\mathbf{2}^+ + a_2\mathbf{3}^-\mathbf{2}^+ + 2a_2\mathbf{4}^-\mathbf{2}^+ - 2a_2\mathbf{5}^-\mathbf{2}^+ + a_3\mathbf{1}^-\mathbf{3}^+\\
&\qquad - a_3\mathbf{2}^-\mathbf{3}^+ - 2a_3\mathbf{5}^-\mathbf{3}^+ - a_5\mathbf{4}^-\mathbf{5}^+\\
0 &= -2a_2 - a_3 - a_5 + d + a_3\mathbf{1}^-\mathbf{3}^+ - a_3\mathbf{2}^-\mathbf{3}^+ - 2a_3\mathbf{4}^-\mathbf{3}^+\\
0 &= -2a_2\mathbf{5}^-\mathbf{2}^+ + 2a_3\mathbf{4}^-\mathbf{3}^+ - 2a_3\mathbf{5}^-\mathbf{3}^+ - 2a_3m^2\mathbf{3}^+ - a_5m^2\mathbf{5}^+
\end{aligned}
$$

e) The implementation can be found online:
`https://gitlab.com/yannickulrich/loop-integration/-/blob/root/code/sheet2.m`