Practical guide to analytic loop integration
Exercise Sheet 1

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Exercise 1: Γ functions

Use our basic integral

$$\int_0^\infty \mathrm{d}x\ x^\alpha (a+bx)^\beta = \frac{\Gamma(1+\alpha)\Gamma(-1-\alpha-\beta)}{\Gamma(-\beta)}\frac{a^{1+\alpha+\beta}}{b^{1+\alpha}} \tag{1}$$

to calculate

a) $I_1(\alpha,\beta,\gamma,\delta) = \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\delta(\cdots)x_1^\alpha x_2^\beta x_3^\gamma \big(x_2x_3+x_1x_2+x_1x_3\big)^\delta$

b) $I_2(\alpha,\beta,\gamma) = \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)x_1^\alpha \big(x_2x_3+x_2x_4+x_3x_4\big)^\beta \big(x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4\big)^\gamma$

c) $I_2(-2\epsilon,-1-2\epsilon,-1+3\epsilon)$ up to $\mathcal{O}(\epsilon^0)$

SOLUTION:

a) The order of integration does not matter in this example. Let us pick x_1, x_2, and finally x_3 with the δ function

$$\begin{aligned}
I_1(\alpha,\beta,\gamma,\delta) &= \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\delta(\cdots)x_1^\alpha x_2^\beta x_3^\gamma \big(x_2x_3+x_1x_2+x_1x_3)\big)^\delta \\
&= \frac{\Gamma(\alpha+1)\Gamma(-\alpha-\delta-1)}{\Gamma(-\delta)}\int \mathrm{d}x_2\mathrm{d}x_3\delta(\cdots)(x_2+x_3)^{-\alpha-1}x_2^{\alpha+\beta+\delta+1}x_3^{\alpha+\delta+\gamma+1} \\
&= \frac{\Gamma(-\alpha-\delta-1)\Gamma(-\beta-\delta-1)\Gamma(\alpha+\beta+\delta+2)}{\Gamma(-\delta)}\underbrace{\int \mathrm{d}x_3\delta(1-x_3)x_3^{\alpha+\beta+2\delta+\gamma+2}}_{1}
\end{aligned}$$

b) This example is slightly more complicated. Right now we cannot integrate eg. x_2. By first solving x_1, we can then solve x_2, and finally use the δ function for either x_3 or x_4

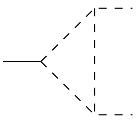
$$
\begin{aligned}
I_2(\alpha,\beta,\gamma) &= \int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)x_1^\alpha \\
&\qquad \big(x_2x_3+x_2x_4+x_3x_4\big)^\beta \big(x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4\big)^\gamma \\
&= \frac{\Gamma(1+\alpha)\Gamma(-1-\alpha-\gamma)}{\Gamma(-\gamma)} \\
&\qquad \int \mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)\big(x_3+x_4\big)^{-1-\alpha}\big(x_2x_3+x_2x_4+x_3x_4\big)^{\beta+1+\alpha+\gamma} \\
&= \frac{\Gamma(\alpha+1)\Gamma(-\alpha-\gamma-1)\Gamma(-\alpha-\beta-\gamma-2)}{\Gamma(-\gamma)\Gamma(-\alpha-\beta-\gamma-1)} \\
&\qquad \int \mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)x_3^{\alpha+\beta+\gamma+2}x_4^{\alpha+\beta+\gamma+2}(x_3+x_4)^{-\alpha-2} \\
&= \frac{\Gamma(\alpha+1)\Gamma(-\alpha-\gamma-1)\Gamma(-\beta-\gamma-1)\Gamma(-\alpha-\beta-\gamma-2)\Gamma(\alpha+\beta+\gamma+3)}{\Gamma(\alpha+2)\Gamma(-\gamma)\Gamma(-\alpha-\beta-\gamma-1)}
\end{aligned}
$$

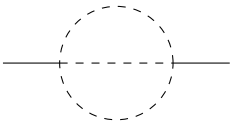
c)

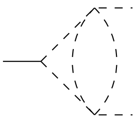
$$
I_2(-2\epsilon,-1-2\epsilon,-1+3\epsilon) = -\frac{1}{\epsilon^2}-\frac{2}{\epsilon}-8+\frac{\pi^2}{2}
$$

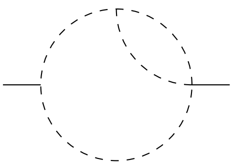
Exercise 2: Graph polynomials

Find the graph polynomials and calculate the loop integrals

a) 

b) 

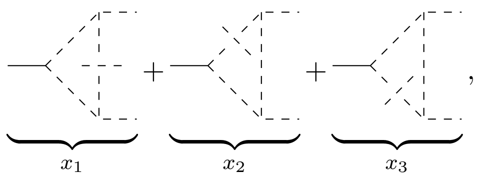
c) 

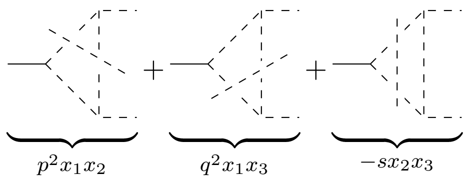
d) 

Dashed lines represent massless propagators and solid lines massive ones.

SOLUTION:

a) We use the graphical method. At $\ell = 1$ this means one cut for $\mathcal{U}$ and two for $\mathcal{F}$

$\mathcal{U} =$ 

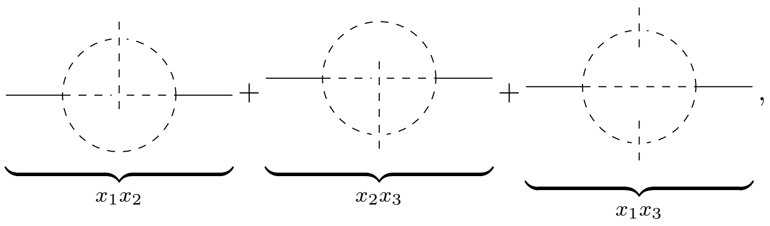
$\mathcal{F} =$ 

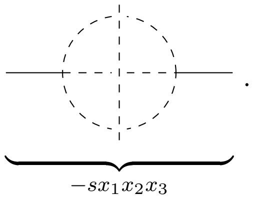
$= -s x_2 x_3 \,.$

Hence,

$$
\begin{aligned}
I_1 &= -\Gamma(1+\epsilon)\Gamma(1-\epsilon) \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \delta(\cdots) \frac{(x_1+x_2+x_3)^{-1+2\epsilon}}{(-s)^{1+\epsilon} x_2^{1+\epsilon} x_3^{1+\epsilon}} \\
&= -(-s)^{-1-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)\Gamma(-\epsilon)^2}{\Gamma(1-2\epsilon)} \,.
\end{aligned}
$$

b) At $\ell = 2$, we need two cuts for $\mathcal{U}$ and three for $\mathcal{F}$

$\mathcal{U} =$ 

$\mathcal{F} =$ 

Hence, the integral is

$$
\begin{aligned}
I_2 &= -\Gamma(1-\epsilon)^2\Gamma(2\epsilon-1)\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\delta(\cdots)\frac{(x_1x_2+x_2x_3+x_1x_3)^{-3+3\epsilon}}{(-s)^{1+2\epsilon}x_2^{1+2\epsilon}x_3^{1+2\epsilon}} \\
&= -(-s)^{1-2\epsilon}\frac{\Gamma(1-\epsilon)^5\Gamma(-1+2\epsilon)}{\Gamma(3-3\epsilon)}\,.
\end{aligned}
$$

c) Using the same methods as above,

$$
\begin{aligned}
\mathcal{U} &= x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4\,, \\
\mathcal{F} &= (-s)x_1x_3x_4 + (-s)x_2x_3x_4\,.
\end{aligned}
$$

We set $\delta(1-x_1)$ and integrate first x_3, then x_4, and finally x_2

$$
\begin{aligned}
I_3 &= \Gamma(1-\epsilon)^2\Gamma(2\epsilon)\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)\frac{(x_1x_2+x_1x_3+x_2x_3+x_1x_4+x_2x_4)^{-2+3\epsilon}}{(-s)^{2\epsilon}(x_1+x_2)^{2\epsilon}x_3^{2\epsilon}x_4^{2\epsilon}} \\
&= (-s)^{-2\epsilon}\frac{\Gamma(1-2\epsilon)^2\Gamma(1-\epsilon)^4\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(2-3\epsilon)\Gamma(2-2\epsilon)}\,.
\end{aligned}
$$

d)

$$
\begin{aligned}
\mathcal{U} &= x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4\,, \\
\mathcal{F} &= (-s)x_1x_2x_3 + (-s)x_1x_2x_4(-s)x_1x_3x_4\,.
\end{aligned}
$$

We set $\delta(1-x_4)$ and integrate first x_1, then x_2, and finally x_3

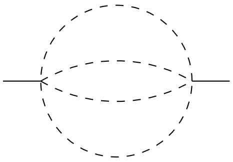
$$
\begin{aligned}
I_4 &= \Gamma(1-\epsilon)^2\Gamma(2\epsilon)\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\delta(\cdots)\frac{(x_1x_3+x_2x_3+x_1x_4+x_2x_4+x_3x_4)^{-2+3\epsilon}}{(-s)^{2\epsilon}x_1^{2\epsilon}(x_2x_3+x_2x_4+x_3x_4)^{2\epsilon}} \\
&= (-s)^{-2\epsilon}\frac{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)^5\Gamma(\epsilon)\Gamma(2\epsilon)}{\Gamma(2-3\epsilon)\Gamma(2-2\epsilon)\Gamma(1+\epsilon)}\,.
\end{aligned}
$$

Exercise 3: Sunset integral

For general masses the sunset integral of 2b cannot be solved at two-loop. However, in the example above with massless internal lines and $p^2 \neq 0$ it was rather easy.

a) This is also true at three loop. Show that

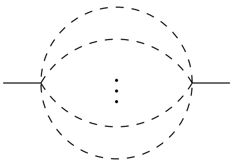
$$I_{\ell=3} =$$



$$= (-p^2)^{2-3\epsilon} \frac{\Gamma(1-\epsilon)^7 \Gamma(-2+3\epsilon)}{\Gamma(4-4\epsilon)} .$$

b) Find a recurrence relation for the ℓ-loop graph polynomials $\mathcal{F}_\ell$ and $\mathcal{U}_\ell$ as a function of $\mathcal{F}_{\ell-1}$ and $\mathcal{U}_{\ell-1}$. Then construct the ℓ-loop sunset I_ℓ, integrate over one Feynman parameter, and relate the result to $I_{\ell-1}$. Finally, solve the resulting recurence relation to obtain

$$I_\ell =$$



$$= (-1)^{\ell+1} (-p^2)^{\ell-1-\ell\epsilon} \frac{\Gamma(1-\epsilon)^{2\ell+1} \Gamma\big(1-\ell(1-\epsilon)\big)}{\Gamma\big((1+\ell)(1-\epsilon)\big)}$$

c) Write code to calculate I_ℓ using only the algorithm for $\mathcal{U}$ and $\mathcal{F}$ and the master formula. *(for the adventurous)*

d) Use the optical theorem to relate $\Im I_\ell$ to the phase space volume for a $1 \to n$ decay

$$\int \mathrm{dPS}^{1\to n} = \frac{s^{n-2}}{2(4\pi)^{2n-3}\Gamma(n)\Gamma(n-1)} .$$

SOLUTION:

a) We have

$$\mathcal{U}_{\ell=3} = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 ,$$
$$\mathcal{F}_{\ell=3} = (-p^2) x_1x_2x_3x_4 .$$

For simplicity we set $p^2 = -1$. Hence,

$$I_{\ell=3} = \Gamma(1-\epsilon)^3 \Gamma(-2+3\epsilon) \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \delta(\cdots) \mathcal{U}^{-4+4\epsilon} \mathcal{F}^{2-3\epsilon} = \frac{\Gamma(1-\epsilon)^7 \Gamma(-2+3\epsilon)}{\Gamma(4-4\epsilon)} .$$

b) From what we have seen we can guess that

$$\mathcal{U}_\ell = \prod_{i=1}^{\ell} x_i + \mathcal{U}_{\ell-1} x_{\ell+1} ,$$
$$\mathcal{F}_\ell = \prod_{i=1}^{\ell+1} x_i$$

Hence, using our master formula with $r = \ell + 1$

$$
\begin{aligned}
I_\ell &= (-1)^{\ell+1}\Gamma(1-\epsilon)^\ell \Gamma\big(1-(1-\epsilon)\ell\big) \int \mathrm{d}x_1 \cdots \mathrm{d}x_{\ell+1}\delta(\cdots)\mathcal{U}^{(1+\ell)(\epsilon-1)}\mathcal{F}^{-1+(1-\epsilon)\ell} \\
&= (-1)^{\ell+1}\Gamma(1-\epsilon)^\ell \Gamma\big(1-(1-\epsilon)\ell\big) \int \left(\prod_{i=2}^{\ell+1} \mathrm{d}x_i\, x_i^{-1+(1-\epsilon)\ell}\right)\left(\prod_{i=2}^{\ell} x_i + \mathcal{U}_{\ell-1}x_{\ell+1}\right)^{(1+\ell)(\epsilon-1)},
\end{aligned}
$$

where we have used $\delta(1-x_1)$. Since $\mathcal{U}_{\ell-1}$ does not depend on $x_{\ell+1}$, we can solve this integration

$$
\begin{aligned}
I_\ell &= (-1)^{\ell+1}\Gamma\big(1-(1-\epsilon)\ell\big)\frac{\Gamma(1-\epsilon)^{\ell+1}\Gamma(\ell-\ell\epsilon)}{\Gamma\big((1-\epsilon)(1+\ell)\big)} \int \left(\prod_{i=2}^{\ell} \mathrm{d}x_i\, x_i^{(1+\ell)(\epsilon-1)}\right)\mathcal{U}_{\ell-1}^{\ell(\epsilon-1)} \\
&= -\Gamma(1-\epsilon)^2 \underbrace{\frac{\Gamma\big(1-\ell(1-\epsilon)\big)}{\Gamma\big((1+\ell)(1-\epsilon)\big)}}_{a_\ell} \underbrace{\frac{\Gamma(\ell-\ell\epsilon)}{\Gamma\big(1+(1-\ell)(1-\epsilon)\big)}}_{1/a_{\ell-1}} I_{\ell-1}\,.
\end{aligned}
$$

This pattern of $a_\ell/a_{\ell-1}$ means that we can telescope the recurrence relation and find

$$
\begin{aligned}
I_\ell &= -\Gamma(1-\epsilon)^2\frac{a_\ell}{a_{\ell-1}}I_{\ell-1} = -(-1)^\ell\Gamma(1-\epsilon)^{2\ell-2}\frac{a_\ell}{a_{\ell-1}}\frac{a_{\ell-1}}{a_{\ell-2}}\cdots\frac{a_2}{a_1}I_1 \\
&= -(-1)^\ell\Gamma(1-\epsilon)^{2\ell-2}\frac{a_\ell}{a_1}I_1\,.
\end{aligned}
$$

With $I_1 = \Gamma(1-\epsilon)^3\Gamma(\epsilon)/\Gamma(2-2\epsilon)$ we arrive at

$$
I_\ell = (-1)^{\ell+1}(-p^2)^{\ell-1-\ell\epsilon}\frac{\Gamma(1-\epsilon)^{2\ell+1}\Gamma\big(1-\ell(1-\epsilon)\big)}{\Gamma\big((1+\ell)(1-\epsilon)\big)}
$$

after restoring the mass dimension of the integral.

c) We have the optical theorem

$$
\int \mathrm{dPS}^{1\to(\ell+1)} = -2\Im I_\ell\,.
$$

Restoring the $1/(16\pi^2)$ per loop order

$$
\int \mathrm{dPS}^{1\to(\ell+1)} = -2\Big(\frac{1}{16\pi^2}\Big)^\ell \Im\left((-s)^{\ell-1-\ell\epsilon}\frac{\Gamma(1-\epsilon)^{2\ell+1}\Gamma\big(1-\ell(1-\epsilon)\big)}{\Gamma\big((1+\ell)(1-\epsilon)\big)}\right)_{\epsilon\to0}.
$$

With $\Im(-s)^{a\epsilon} \sim a\pi\epsilon$ and $\Gamma(1-\ell(1-\epsilon)) \sim (-1)^{\ell+1}/\epsilon/\Gamma(1+\ell)$

$$
\int \mathrm{dPS}^{1\to(\ell+1)} = 2\Big(\frac{1}{16\pi^2}\Big)^\ell \ell\pi\epsilon(-s)^{\ell-1}\frac{(-1)^{\ell+1}}{\epsilon\Gamma(1+\ell)^2} = \frac{s^{\ell-1}}{2(4\pi)^{2\ell-1}\Gamma(1+\ell)\Gamma(\ell)}
$$