Practical guide to analytic loop integration
Exercise Sheet 3

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https://yannickulrich.gitlab.io/loop-integration
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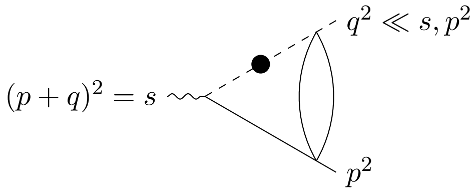
Exercise 1: top quark decay

Consider the following integral that appears in the two-loop calculation of the top decay



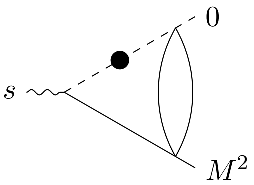
(1)

The dashed line corresponds to light b quark and the solid line to the heavy t quark.

a) Find the momentum regions that contribute to this process.

b) Show that the hard region does in fact equal the same calculation with massless b quarks, i.e. $m = 0$.

c) The result for $m = 0$ can be found in the literature as

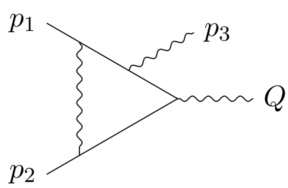


$$= \frac{(M^2)^{-2\epsilon}}{M^2 - s}\Bigg[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\frac{1+y}{y}H_1(y) + 4 + \frac{2(1+y)}{y}\Big(H_1(y) + H_{1,1}(y)\Big) + \frac{2}{y}\frac{1+y}{1-y}\Big(H_{0,1}(y) - y\zeta_2\Big)\Bigg]. \tag{2}$$

Show by explicit calculation of the remaining regions that the complete integral has no $1/\epsilon^2$ pole but a $\log(m^2/M^2)$.

Exercise 2: Soft approximation

Consider the following diagram



$$= \int [\mathrm{d}k] \frac{1}{\big[k^2\big]\big[(k-p_1)^2 - m^2\big]\big[(k-p_1+p_3)^2 - m^2\big]\big[(k+p_2)^2 - m^2\big]} \tag{3}$$

with the energy of the photon soft compared to $s = Q^2$ and $p_1^2 = p_2^2 = m^2 \ll s$. This can be phrased in a Lorentz invariant way by requiring that all invariants

$$\sigma_{i3} = 2p_i \cdot p_3 \sim m \sim \lambda\,. \tag{4}$$

a) Use the method of regions in the parametric representation to find all six regions that contribute to this integral.

b) Consider the region $\vec{r}^{(1)} = (0, -1, -1, 1)$, i.e. $\mathcal{P}_1 \sim 1$, $\mathcal{P}_2 \sim \lambda^{-1}$, $\mathcal{P}_3 \sim \lambda^{-1}$, $\mathcal{P}_4 \sim \lambda$. Show that this integral is not finite in dimensional regularisation.

Such behaviour is not uncommon and usually points to a broken symmetry eg. in SCET. It is usually addressed by using analytic regularisation, i.e.

$$\begin{aligned}&\int [\mathrm{d}k] \frac{1}{\big[k^2\big]\big[(k-p_1)^2-m^2\big]\big[(k-p_1+p_3)^2-m^2\big]\big[(k+p_2)^2-m^2\big]}\\ \to \big(-\nu^2\big)^\eta &\int [\mathrm{d}k] \frac{1}{\big[k^2\big]\big[(k-p_1)^2-m^2\big]\big[(k-p_1+p_3)^2-m^2\big]^{1+\eta}\big[(k+p_2)^2-m^2\big]}\,.\end{aligned} \tag{5}$$

We have introduced an additional regulator η that we will take to zero as soon as we have added all regions. Crucially, $\eta \to 0$ needs to be done *before* $\epsilon \to 0$.

c) Calculate the integral, up to $\mathcal{O}(\lambda^0)$. Add all regions and set $\eta \to 0$ and finally $\epsilon \to 0$.