

# Practical guide to analytic loop integration

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version cb59ba9

## Abstract

Loop integration is a vital part of any higher order calculation. However, practical tools to actually compute these integrals are rarely covered in lectures. In this block course, I will cover some advanced methods that have been used in a number of multi-scale multi-loop calculations.

After introducing the problem, I will discuss integration-by-parts reduction to reduce the number of integrals, the method of regions to reduce the number of scales and finally the Mellin Barnes technique to actually solve the integrals.

The course will be composed of lectures introducing the techniques and practical, hands-on example at the two-loop level.

For further reading, see [1, 2, 3, 4, 5, 6].

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# 1 Basics of loop integration

The calculation of Feynman integrals is a core problem in the calculation of higher order problems. Over the last decades, many techniques have been developed, both analytic and numeric. Many resources are available for commonly used techniques such as differential equations or sector decomposition. In this course, we will cover some more niche techniques that are useful for specialist application.

**Definition 1** (Scalar loop integral). We will almost exclusively consider integrals of form

$$\int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_\ell}{(2\pi)^d} \frac{1}{\mathcal{P}_1^{\alpha_1} \cdots \mathcal{P}_t^{\alpha_t}}, \quad (1)$$

with  $\ell$  loop momenta  $k_i$ . The powers  $\alpha_i$  of the propagators  $\mathcal{P}_i$  may be positive, negative, or zero. The propagators themselves are either linear or quadratic in the loop momenta. For example,

$$\mathcal{P}_i = \begin{cases} (k_i + p_i)^2 - m_i^2 + i0^+ \\ 2k_i \cdot p_i - m_i^2 + i0^+ \end{cases}, \quad (2)$$

where  $i0^+$  denotes the small imaginary part of the Feynman prescription required to ensure causality.

**Definition 2** (Loop measure). When solving loop integrals, we often encounter series expansions of the  $\Gamma$  function. This means that our intermediary results for the virtual matrix elements will contain terms proportional to the Euler constant  $\gamma_E = -\Gamma(1)' = 0.5772\dots$ . This is completely unphysical and will cancel once the virtual and real corrections are combined. It is, however, convenient to already drop these at the integral level. Hence, we re-define the loop measure as

$$[dk] = \Gamma(1 - \epsilon) \mu^{2\epsilon} \frac{d^d k}{i\pi^{d/2}}. \quad (3)$$

In what follows, I will often set  $\mu = 1$  unless I want to make a specific point. Other definitions are used in the literature. If you are reusing results by other people make sure to check the conventions they have used.

**Theorem 3** (Tadpole integral). For  $\ell = t = 1$  we have the tadpole integral

$$\int [dk] \frac{1}{[k^2 - m^2 + i0^+]^n} = (-1)^n (m^2)^{2-n-\epsilon} \frac{\Gamma(1 - \epsilon)\Gamma(n - 2 + \epsilon)}{\Gamma(n)} \quad (4)$$

*Proof.* Our integrand has poles at  $k_0 = \pm\sqrt{\vec{k}^2 + m^2} \mp i0^+$  as shown in Figure 1. Since the poles are in the top-left and lower-right quadrant, the integral over the drawn contour vanishes. Since the integrand falls quickly enough, the integrals over the real and imaginary axis are equal up to a sign

$$0 = \oint dk_0 = \left( \int_{-\infty}^{\infty} + \int_{-\infty}^{i\infty} + \text{arcs} \right). \quad (5)$$

Substituting  $k_0 \rightarrow ik_0$  so that  $k^2 = -k_0^2 - \vec{k}^2 = -k_E^2$  where  $k_E$  is a Euclidean momentum. This process is known as a Wick rotation, transforming our expression to

$$\begin{aligned} \int [dk] \frac{1}{[k^2 - m^2 + i0^+]^n} &= \frac{\Gamma(1 - \epsilon)}{\pi^{d/2}} (-1)^n \int d^d k_E \frac{1}{[k_E^2 + m^2 + i0^+]^n} \\ &= \frac{\Gamma(1 - \epsilon)}{\pi^{d/2}} (-1)^n \int d\Omega_d \int_0^\infty dk_E \frac{k_E^{d-1}}{[k_E^2 + m^2]^n}. \end{aligned} \quad (6)$$

At this point we can set  $i0^+ = 0$  since the denominator is always strictly bigger than zero. Using that the  $d$ -dimensional sphere has a volume of

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \quad (7)$$

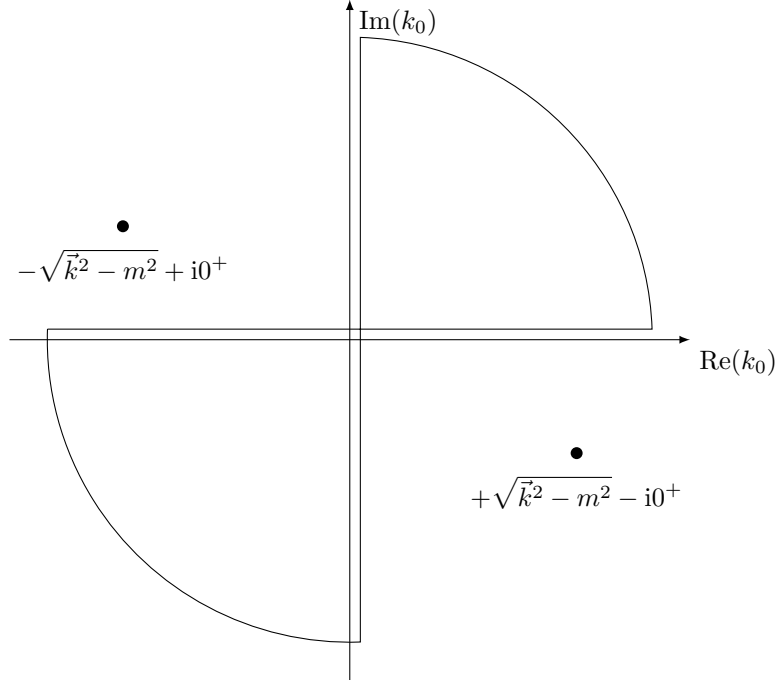


Figure 1: The poles of the  $k_0$  intergation and the contour of (5)

and the basic integral (which is nothing but the definition of the  $B$  function)

$$\int_0^\infty dx x^\alpha (a + bx)^\beta = \frac{\Gamma(1 + \alpha)\Gamma(-1 - \alpha - \beta)}{\Gamma(-\beta)} \frac{a^{1+\alpha+\beta}}{b^{1+\alpha}}, \quad (8)$$

we find

$$\begin{aligned} \int [dk] \frac{1}{[k^2 - m^2 + i0^+]^n} &= \frac{\Gamma(1 - \epsilon)}{\pi^{d/2}} (-1)^n \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} (m^2)^{d/2-n} \frac{\Gamma(\frac{d}{2})\Gamma(n - \frac{d}{2})}{2\Gamma(n)} \\ &= (-1)^n (m^2)^{2-n-\epsilon} \frac{\Gamma(1 - \epsilon)\Gamma(n - 2 + \epsilon)}{\Gamma(n)}. \end{aligned} \quad (9)$$

□

To solve any loop integral with more propagators than loops, we could either solve the loops one by one or all in one go. Both methods are equivalent though difficult to relate in practical examples. We will be focusing on the latter case as the former can be viewed as a sub-class.

**Lemma 4** (Feynman parametrisation & Cheng-Wu theorem [7]). We write

$$\frac{1}{\mathcal{P}_1^{\alpha_1} \dots \mathcal{P}_t^{\alpha_t}} = \frac{\Gamma(r)}{\prod_j \Gamma(\alpha_j)} \int_0^\infty \prod_{j=1}^t dx_j x_j^{\alpha_j-1} \delta\left(\sum_{i \in \nu} x_i - 1\right) \frac{1}{(\mathcal{P}_1 x_1 + \dots + \mathcal{P}_t x_t)^r}. \quad (10)$$

where  $r = \sum_j \alpha_j$  and  $\nu$  a non-empty subset of  $\{1, \dots, t\}$  (Cheng-Wu theorem).

Note that most books on QFT will assume  $\nu = \{1, \dots, t\}$ , reducing the integration region to  $[0, 1] \times [1, 1 - x_1] \times \dots$ . However, for analytic calculations we have found that having just one element, say  $i = 1$ , in  $\nu = \{1\}$  is a better choice, setting one  $x_1 = 1$  and keeping the integration bounds at  $[0, \infty]^{t-1}$ .

**Theorem 5.** We can calculate a general  $\ell$ -loop integral in terms of graph polynomials or Symanzik polynomials

$$I = (-1)^r \Gamma(1 - \epsilon)^\ell \frac{\Gamma(r - \ell d/2)}{\prod_j \Gamma(\alpha_j)} \int_0^\infty \prod_{j=1}^t dx_j x_j^{\alpha_j-1} \delta(x_i - 1) \underbrace{\frac{\mathcal{U}^{r-(\ell+1)d/2}}{(\mathcal{F} - i0^+)^{r-\ell d/2}}}_{\mathcal{G}}. \quad (11)$$

$\mathcal{U}$  and  $\mathcal{F}$  are the graph polynomials or Symanzik polynomials that can be computed analytically (see (17)), algorithmically (see Observation 6), or graph-theoretically (see Observation 7).

Further,  $\mathcal{U} > 0$  so its prescription does not matter.  $\mathcal{F}$ , however, can be both positive and negative. It is hence important to properly include its prescription as  $\mathcal{F} \rightarrow \mathcal{F} - i0^+$ .

*Proof.* We use (10) to write the dominator as

$$D = \mathcal{P}_1 x_1 + \dots + \mathcal{P}_t x_t = k^T \cdot M(x_i) \cdot k - 2Q(x_i, q_j)^T k + J(x_i, s_{jk}) + i0^+, \quad (12)$$

with a  $\ell \times \ell$  matrix  $M$ ,  $\ell$ -vectors  $k = (k_1, \dots, k_\ell)$  and  $Q$ , depending on the Feynman parameters  $x_i$ , external momenta  $q_j$  and invariants  $s_{jk} = 2q_j \cdot q_k$ . By shifting  $k \rightarrow k + M^{-1}Q$  we cancel the linear term so that after diagonalising  $M$  (with eigenvalues  $\lambda_i$ ) we have

$$D = k^T \cdot \text{diag}(\lambda_i) \cdot k - \Delta \quad \text{with} \quad \Delta = Q^T M^{-1} Q - J - i0^+. \quad (13)$$

$\Delta$  has a  $-i0^+$  prescription since we started with  $J + i0^+$  in (12). Next, we rescale  $k_i \rightarrow \lambda_i^{-1/2} k_i$  to factorise the loop integrations

$$I = \frac{\Gamma(r)}{\prod_j \Gamma(\alpha_j)} \int_0^\infty \prod_{j=1}^t dx_j x_j^{\alpha_j-1} \delta(\dots) \underbrace{\int \lambda_1^{-d/2} [dk_1] \dots \lambda_\ell^{-d/2} [dk_\ell] \frac{1}{[k^T \cdot k - \Delta]^r}}_{I'} \quad (14)$$

We can extend (4) to cover this case

$$\begin{aligned} I' &= \left( \frac{\Gamma(1-\epsilon)}{\pi^{d/2}} \right)^\ell (-1)^r \int d^d k_{1,E} \dots d^d k_{\ell,E} \frac{1}{[k_{1,E}^2 + \dots + k_{\ell,E}^2 + \Delta]^r} \\ &= (-1)^r \Delta^{\ell d/2 - r} \frac{\Gamma(1-\epsilon)^\ell \Gamma(r - \ell d/2)}{\Gamma(r)}. \end{aligned} \quad (15)$$

With this, we can write

$$I = \frac{\Gamma(r)}{\prod_j \Gamma(\alpha_j)} \int_0^\infty \prod_{j=1}^t dx_j x_j^{\alpha_j-1} \lambda_i^{-d/2} \delta(\dots) (-1)^r \Delta^{-r + \ell d/2} \frac{\Gamma(1-\epsilon)^\ell \Gamma(r - \ell d/2)}{\Gamma(r)}. \quad (16)$$

Here we have used (4) to find the general Feynman-parametrised form of the  $\ell$ -loop integral. We now identify

$$\mathcal{U} = \det M = \prod_j \lambda_j, \quad \mathcal{F} = \det M \times \Delta, \quad (17)$$

to arrive at (11). □

**Observation 6** (Algorithmic calculation of  $\mathcal{U}$  and  $\mathcal{F}$ ). We can calculate  $\mathcal{U}$  and  $\mathcal{F}$  directly without having to manually find the eigenvalues  $\lambda_i$ . Instead, we calculate  $\Delta$  and  $\det M$  directly from (12). Collecting terms of  $k_1$

$$(12) \equiv D_1 = k_1^2 t_1^{(2)} + k_1 \cdot t_1^{(1)} + t_1^{(0)}, \quad (18)$$

we identify  $t_1^{(2)} \equiv \lambda_1$  and set

$$D_2 = t_1^{(0)} - \frac{t_1^{(1)} \cdot t_1^{(1)}}{4t_1^{(2)}}. \quad (19)$$

We now repeat this and identify  $t_2^{(i)}$

$$D_2 = k_2^2 t_2^{(2)} + k_2 \cdot t_2^{(1)} + t_2^{(0)} \quad (20)$$

in order to construct  $D_3$ . Eventually, we will reach  $D_{\ell+1} = -\Delta$ . This can be efficiently implemented in Mathematica as shown in Listing 2.

```

(* This function takes D_i and U_i == \lambda_1 * ... * \lambda_i and
returns D_{i+1} and U_{i+1} *)
UFstep[{di_, ui_}, k_] := Module[{t0, t1, t2},
  t0 = Coefficient[di, k, 0];
  t1 = Coefficient[di, k, 1];
  t2 = Coefficient[di, k, 2];

  {t0 - t1*t1 / (4*t2), ui * t2}
]

(* This functions takes a list of loop momenta and propagators and
folds UFstep over it *)
UF[ks_, props_] := Module[{d0, d1, ul},
  d0 = Total[props x /@ Range[Length[props]]];
  {d1, ul} = Fold[UFstep, {d0, 1}, ks];
  {ul, d1 ul}
]

```

Listing 2: Mathematica implementation of  $\mathcal{U}$  and  $\mathcal{F}$

**Observation 7** (Graph-theoretical determination of  $\mathcal{U}$  and  $\mathcal{F}$ ).  $\mathcal{U}$  and  $\mathcal{F}$  can also be determined just by considering the graph of the loop integral in question. (12) assigns a Feynman parameter  $x_i$  to each edge of the graph. For example, consider the following one-loop integral

$$\begin{array}{c}
\begin{array}{c} \xrightarrow{p} \\ \text{---} \\ \begin{array}{c} x_2 \\ \diagdown \\ \text{---} \\ \diagup \\ x_3 \end{array} \\ \text{---} \\ \xrightarrow{q} \end{array} \\
= \int [dk_1] \frac{1}{k_1^2} \frac{1}{(k_1 + p)^2 - m^2} \frac{1}{(k_1 - q)^2 - m^2}
\end{array} \quad (21)$$

$\mathcal{U}$  is then determined by adding the products  $x_i$  for each  $\ell$  line cuts of propagators that results in a tree-level

$$\begin{array}{c}
\mathcal{U} = \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} \\
= x_1 + x_2 + x_3 .
\end{array} \quad (22)$$

Similarly, for massless propagators,  $\mathcal{F}$  is found by adding all  $(\ell + 1)$  line cuts with the momentum flowing through the resulting tree

$$\begin{array}{c}
\mathcal{F}_0 = \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} \\
= p^2 x_1 x_2 + q^2 x_1 x_3 + (p + q)^2 x_2 x_3 .
\end{array} \quad (23)$$

To account for massive propagators we write

$$\mathcal{F} = \mathcal{F}_0 + \sum x_i m_i^2 \mathcal{U}, \quad (24)$$

where  $m_i$  is the mass of the propagator associated to  $x_i$ .

**Theorem 8** (Master formula for loop integrals). An arbitrary integral, including numerators, can be

calculated as

$$I = (-1)^r \Gamma(1 - \epsilon)^\ell \Gamma(r - s - \ell d/2) \int_0^\infty \delta(x_i - 1) \left( \prod_{j=1}^t dx_j \frac{x_j^{\alpha_j - 1}}{\Gamma(\alpha_j)} \right) \left( \prod_{j=t+1}^p \frac{\partial^{-\alpha_j}}{\partial x_j^{-\alpha_j}} \right) \frac{\mathcal{U}^{r-s-(\ell+1)d/2}}{(\mathcal{F} - i0^+)^{r-s-\ell d/2}} \Big|_{x_{t+1}=\dots=x_p=0}, \quad (25)$$

with  $r$  ( $s$ ) the sum of positive (negative) indices,  $t$  the number of positive indices and  $p$  the length of the family as defined above. This implies that

$$\alpha_1, \dots, \alpha_t > 0 \quad \text{and} \quad \alpha_{t+1}, \dots, \alpha_p \leq 0. \quad (26)$$

*Proof.* As discussed above, we implement numerators in integrals by setting some  $\alpha_i < 0$ . However, that would make the Feynman parametrisation ill-defined because the  $\Gamma$  function diverges for negative integers. To solve this problem, we note an identity for Mellin transforms called Ramanujan's master theorem. In our language it states that the Mellin transform of a function  $f(x)$  evaluated at negative integers  $-n$  can be written as the  $n$ -th derivative of  $f$

$$\{\mathcal{M}f\}(-n) = \int dx x^{-n-1} f(x) = \Gamma(-n) f^{(n)}(0). \quad (27)$$

Now the  $\Gamma(-n)$  cancels, finally leading to our master formula.  $\square$

## 2 Integration-by-parts reduction

Real world calculations often involve many hundreds of integrals; computing these one by one is clearly infeasible. Instead we require a method to reduce the number of integrals to a manageable number of so-called master integrals that we can then calculate.

### 2.1 Organising integrals

**Definition 9** (Integral family). If we have  $\rho$  independent external momenta (after applying momentum conservation), we have

$$\binom{\binom{\rho}{2}}{2} = \binom{\rho+1}{2} = \frac{(\rho+1)\rho}{2} \quad (28)$$

possible ways to build scalar products (including masses  $p_i^2$ ). Here, we have defined the multichoose function

$$\binom{\binom{n}{k}}{k} = \binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}, \quad (29)$$

that counts the number of ways one can pick  $k$  unordered elements from a set of  $n$  elements, allowing for repetition.

If we have  $\ell$  loop momenta, there are

$$p = \binom{\binom{\rho+\ell}{2}}{2} - \binom{\binom{\rho}{2}}{2} = \ell \frac{1+\ell+2\rho}{2} \quad (30)$$

possible scalar products of involving at least one loop momentum. A set of propagators of this size that allows all scalar products to be written through propagators is called a family. These propagators rarely all belong to the actual diagram and some might be fictitious, added purely to bring up the numbers.

**Definition 10** (Reducible scalar integral). When calculating matrix elements, we often have to find integrals with numerators. Since the family is complete, we can always write these as propagators and bring the matrix element into the following form

$$\sum_n C_n \times \int \prod_{j=1}^{\ell} [dk_j] \frac{1}{\mathcal{P}_{1,n}^{\alpha_{1,n}} \cdots \mathcal{P}_{p,n}^{\alpha_{p,n}}}. \quad (31)$$

```
(* We define t as the number of propagators,
      r as the sum of pos. powers,
      s as the sum of neg. powers *)
TRS[a_List] := {
  Length@Select[a, # > 0 &],
  Total[ Select[a, # > 0 &]],
  Total[-Select[a, # < 0 &]]
}
```

Listing 3: Mathematica implementation of  $t$ ,  $r$ , and  $s$  given an integral

The powers  $\alpha_i$  of the propagators  $\mathcal{P}_i$  may be negative or zero and the  $C_n$  are functions of the external kinematics and the dimension  $d$ . These integrals are referred to as reducible scalar integrals.

If we have scalar products with other momenta or even vector integrals we first have to use Passarino-Veltman decomposition as we would at one-loop.

**Definition 11** (reduce organisation [8]). In virtually no case are all  $\alpha_i > 0$ . Hence, for a given scalar integral we define

- $t$ : the number of  $\alpha_i > 0$
- $r = \sum_{\alpha_i > 0} \alpha_i$  the sum of denominator powers
- $s = - \sum_{\alpha_i < 0} \alpha_i$  the sum of numerator powers
- the sector ID

$$\text{ID} = \sum_{k=1}^t 2^{i_k-1} \quad \text{with} \quad \alpha_{i_1}, \dots, \alpha_{i_t} > 0. \quad (32)$$

Obviously  $s \geq 0$  and  $r \geq t$ . This serves to organise integrals because as soon as one integral in a sector can be calculated all integrals of the sector can be calculated, at least in principle.

**Definition 12** (Corner integral). For a given sector, we call an integral that has only unit powers the corner integral of that sector. It obviously has  $r = t$ .

## 2.2 Seed identities

To reduce the number of scalar integrals, we are using integration-by-parts (IBP) identities

$$\int dx u v' = u v - \int dx u' v. \quad (33)$$

**Theorem 13** (IBP for loop integral). Since the loop integration goes from  $-\infty$  to  $\infty$ , one can show that the surface term  $u v$  vanishes in dimensional regularisation. In the language of loop integrals

$$\int \prod_{j=1}^{\ell} [dk_j] \frac{\partial}{\partial k_i} \cdot \left( q \frac{1}{\mathcal{P}_1^{\alpha_1} \dots \mathcal{P}_t^{\alpha_t}} \right) = 0, \quad i = 1, \dots, \ell, \quad (34)$$

where  $q$  represents either a loop or an external momentum. Note that  $q$  is inside the derivative s.t. if  $q = k_i$  the product rule has to be used on the integrand with

$$\frac{\partial}{\partial k} \cdot k \equiv \frac{\partial}{\partial k_\mu} k^\mu = d. \quad (35)$$

IBP relations now allow us to get identities between different integrals.

```

moms = {LTensor[k1, mu], LTensor[p, mu], LTensor[q, mu]};

ibps = Collect[ToFamily@Expand[
  D[moms FromFamily[int[{a1, a2, a3}]], k1]
]/. onshell, _int, Factor];

```

Listing 4: Mathematica implementation of the generation of seed identities

**Observation 14** (Heavy quark bubble). Consider the following example

$$I(a, b) = \int [dk] \frac{1}{[k^2]^a [(k-p)^2 - M^2]^b} \quad \text{with } p^2 = M^2. \quad (36)$$

We can relate integrals with different  $a$  and  $b$  through

$$\begin{aligned} q = k: \quad 0 &= (d - 2a - b)I(a, b) - bI(a - 1, b + 1), \\ q = p: \quad 0 &= (-a + b)I(a, b) - bI(a - 1, b + 1) + aI(a + 1, b - 1) + 2bM^2 I(a, b + 1). \end{aligned} \quad (37)$$

**Definition 15** (Seed identity & compact notation). We call identities of the form (37) seed identities because we will be using them to create more identities by choosing  $a$  and  $b$ .

We can write seed identities using the short-hand notation of [2].  $\mathbf{n}^\pm$  indicates that the power of the  $n$ -th propagator is raised (lowered) by one.

$$\begin{aligned} 0 &= d - 2a - b - b \mathbf{1}^- \mathbf{2}^+, \\ 0 &= -a + b - b \mathbf{1}^- \mathbf{2}^+ + a \mathbf{2}^- \mathbf{1}^+ + 2bM^2 \mathbf{2}^+. \end{aligned} \quad (38)$$

*Proof of Observation 14.* Setting  $q = k$ , we apply (34)

$$\begin{aligned} i(a, b) &= \frac{\partial}{\partial k_\mu} \left( k^\mu \frac{1}{[k^2]^a [(k-p)^2 - M^2]^b} \right) \\ &= k^\mu \frac{\partial}{\partial k_\mu} \left( \frac{1}{[k^2]^a [(k-p)^2 - M^2]^b} \right) + \left( \frac{\partial}{\partial k_\mu} k^\mu \right) \frac{1}{[k^2]^a [(k-p)^2 - M^2]^b} \\ &= -2k^\mu \left( \frac{a k^\mu}{[k^2]^{1+a} [(k-p)^2 - M^2]^b} + \frac{b(k^\mu - p^\mu)}{[k^2]^a [(k-p)^2 - M^2]^{1+b}} \right) \\ &\quad + \frac{d}{[k^2]^a [(k-p)^2 - M^2]^b} \\ &= \frac{(d - 2a - 2b)k^2 + 2(2a + b - d)k \cdot p}{[k^2]^a [(k-p)^2 - M^2]^{b+1}}. \end{aligned} \quad (39)$$

We now turn this expression back into scalar integrals of the form  $I(a', b')$ . After loop integration and setting  $\int [dk] i(a, b) = 0$  we finally have our first seed identity

$$0 = -bI(a - 1, b + 1) + (d - 2a - b)I(a, b). \quad (40)$$

With  $q = p$ , we find the other identity. □

### 2.3 Laporta algorithm

We now can use identities like the ones above to reduce integrals. However, doing this by hand is not practical. Instead, we use a modified version of Gaussian elimination to solve a very large though sparse system of linear equations by focussing on the difficult integrals first.



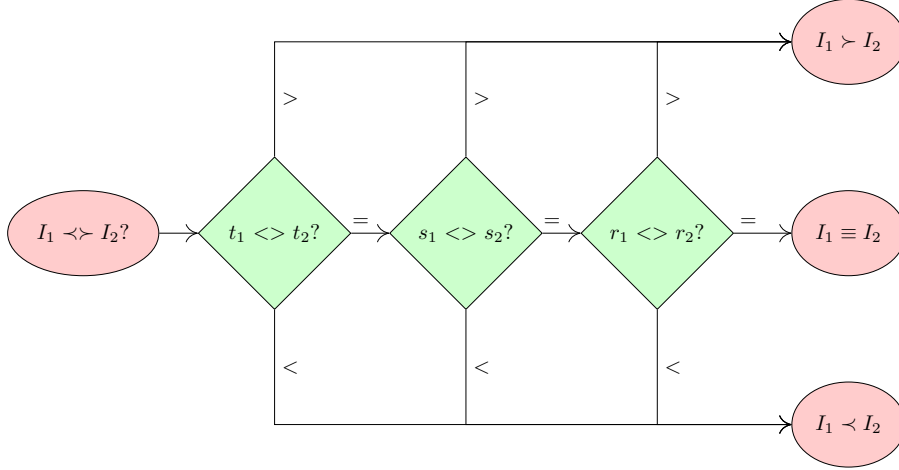


Figure 5: A common lexicographical ordering

**Definition 16** (Lexicographic ordering). Given two integrals, we need a way to decide which one is more complicated. The exact specification of this ordering does not matter as long as it is consistent. We will be using the ordering of Figure 2.3: given  $I_1(t_1, r_1, s_1)$  and  $I_2(t_2, r_2, s_2)$ , it prefers small  $t$ ,  $s$ , and  $r$  in that order.

**Theorem 17** (Laporta's algorithm). We can now reduce a set of irreducible integral to a (hopefully) small set of master integrals using Laporta's algorithm [9, 10] (Figure 6 and Listing 7)

1. Generate  $i_{\max}$  seed identities of Definition 15
2. List all  $j_{\max}$  reducible integrals up to some cut-off  $r_{\max}$  and  $s_{\max}$  and order them using the lexicographical ordering
3. Apply an identity to an integral and call the resulting relation  $\mathcal{R}$
4. Substitute all known relations into  $\mathcal{R}$ , obtaining a relation  $\mathcal{R}' = \sum_i c_i I_i = 0$
5. If  $\mathcal{R}'$  is trivial, i.e.  $c_i = 0$  and hence  $0 = 0$ , go back to Step 3. Otherwise, solve  $\mathcal{R}'$  for the most complicated integral and remember this relation.
6. Go back to Step 3 until all integrals and seed identities have been used.
7. Re-substitute and clean up the relations

This algorithm converges because the cut-off points  $r_{\max}$  and  $s_{\max}$ . The system is naturally over-determined because the number of new integrals grows slower than the number of equations, resulting in many trivial relations along the way.

**Observation 18.** The implementation provided here is only for educational purposes as it is horribly inefficient. Instead, one should use one of many public codes such as AIR [11], FIRE [12], Kira [13], or `reduze` [8]. These codes essentially all implement Laporta's algorithm albeit much cleverer. For example, Kira uses finite-field sampling to improve the performance, especially for the back-substitution which can quickly become a bottleneck. Still, the IBP reduction is often a major bottleneck of higher order calculations and can run for months on a large cluster.

### 3 Method of regions

In general, the calculation of master integrals with full dependence of any parameter is very difficult and time consuming. However, in many cases this is not needed, often because the parameters have a

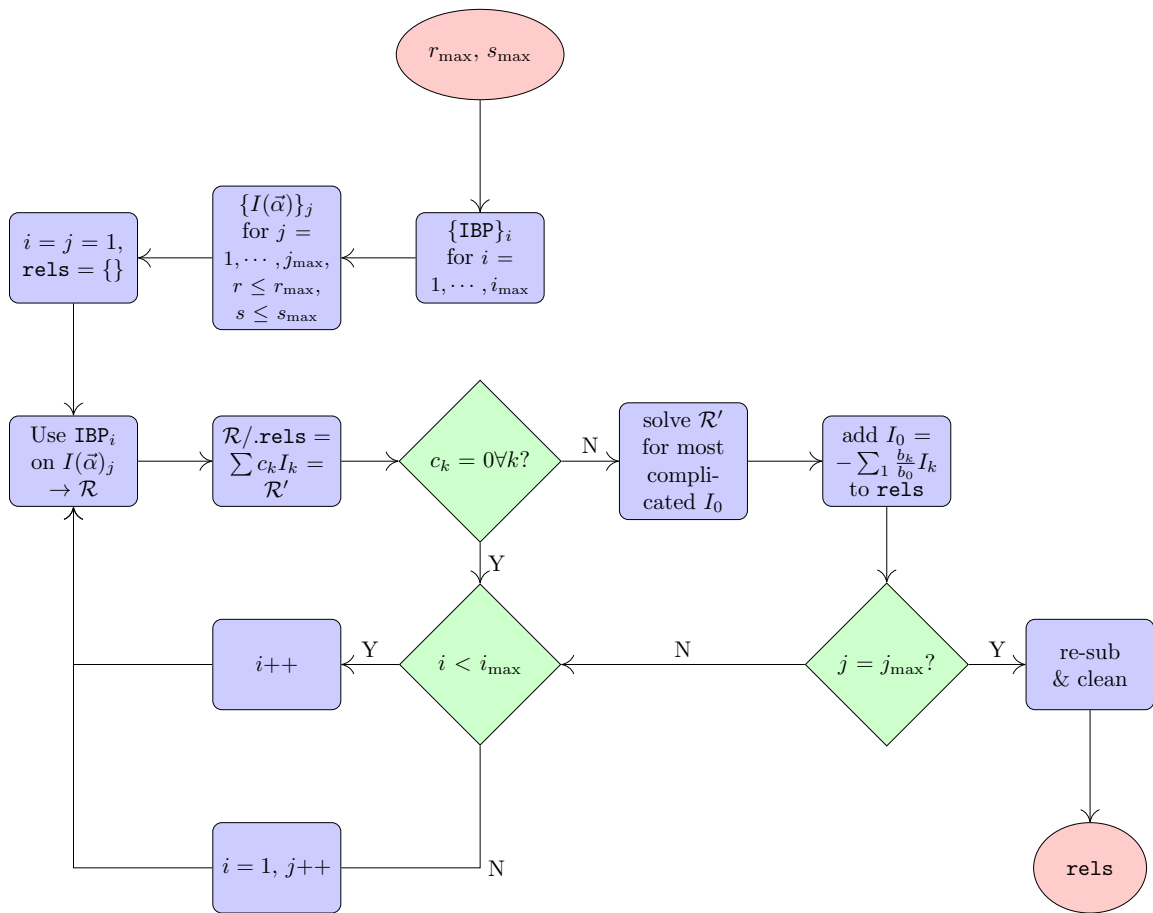


Figure 6: Flowchart for Laporta's algorithm

```

rmax = 5;
smax = 3;
IsOkay[{t_, r_, s_}] /; And[
    t > 0,
    rmax >= r,
    smax >= s
] = True;
IsOkay[_] = False;

intlist = Flatten[Outer[
    int[{-##}]&,
    Sequence @@ ConstantArray[Range[-smax, rmax], Length[family]]
]];

seedPre = Sort[
    Select[intlist, IsOkay @* TRS],
    LexiOrdered
];

seedRep = seedPre /. int[a_] :> ReplaceAll[Thread[(ToExpression["a"<>
    ↪ ToString[#1]] &/@ Range[Length[family]]) -> a]];

(* We now run Laporta's algorithm *)
rels = {i_int /; MemberQ[zerosectors, Sector[i]] -> 0};
Table[
    rel = Collect[seed[id] //. rels, _int, Factor];
    If[rel != 0,
        rels = Flatten[Join[rels, Solve[rel == 0, MostComplicated[rel
            ↪ ]]]];
    ],
    {seed, seedRep[[ ; ]]}, {id, ibps}
];

(* At the borders of the seed range we might get some useless
   identities that relate two complicated integrals we don't know. *)
IsUseless[a_ -> b_] := Quiet[
    LexiOrdered[MostComplicated[a], MostComplicated[b]] === 0
]
RemoveUseless[rels_] := Select[rels, Not@*IsUseless]

(* We also need to do the re-substitution and simplification *)
rels2 = RemoveUseless@Thread[
    rels[[2 ;, 1]] -> Collect[
        rels[[2 ;, 1]] //. rels, _int, Factor
    ]
];

```

Listing 7: Mathematica implementation of Laporta's algorithm

strong hierarchy. This could either mean that the electron mass  $m^2$  is much smaller than the momentum transfer  $Q^2$  or that the  $W$  boson mass  $m_W^2$  is much larger. In these cases, we instead calculate the integrals expanded in the small parameter ( $m/Q^2$  or  $Q^2/m_W^2$ ). The technique used to achieve this is the method of regions [14].

### 3.1 Momentum space

**Observation 19** (Heavy particle). Let us first consider the case of a heavy particle  $m_X^2 \gg s$  in

$$I = \int [dk] \frac{1}{[k^2 - m_X^2 + i0^+][k^2 - 2k \cdot p + i0^+]} \quad (41)$$

with  $p^2 = m^2 \ll m_X^2$ . The naive expansion of the integrand

$$\frac{1}{[k^2 - m_X^2 + i0^+][k^2 - 2k \cdot p + i0^+]} = -\frac{1}{m_X^2} \frac{1}{[k^2 - 2k \cdot p + i0^+]} + \mathcal{O}\left(\frac{1}{m_X^4}\right) \quad (42)$$

is obviously wrong since integration and expansion does not commute. The expansion is wrong in the region where  $k \sim m_X$ .

**Theorem 20** (The method of regions in momentum space). We can expand an integral already at integrand level by identifying all relevant regions, expanding in those up to whatever power we desire and adding them up. In principle infinitely many such regions exist, but most of them vanish in dimensional regularisation.

*Proof.* We only prove the example, though this argument can be formalised.

We split the integration at the scale  $\Lambda$  which is  $m \ll \Lambda \ll m_X$

$$\begin{aligned} I &= \int_0^\Lambda [dk] \frac{1}{[k^2 - m_X^2][k^2 - 2k \cdot p]} + \int_\Lambda^\infty [dk] \frac{1}{[k^2 - m_X^2][k^2 - 2k \cdot p]} \\ &= \underbrace{-\frac{1}{m_X^2} \int_0^\Lambda [dk] \frac{1}{[k^2 - 2k \cdot p]}}_{\text{soft}} + \underbrace{\int_\Lambda^\infty [dk] \left( \frac{1}{[k^2 - m_X^2][k^2]} + \frac{4(k \cdot p)^2}{[k^2 - m_X^2]^3 [k^2]} \right)}_{\text{hard}} + \mathcal{O}\left(\frac{1}{m_x^4}\right) \end{aligned} \quad (43)$$

The first term is generally referred to the soft region and the second term as the hard region. We now need to relate these integrals back to what we can calculate. It turns out we can just integrate over the whole range since the difference is scaleless, eg.

$$\int_\Lambda^\infty [dk] \frac{1}{[k^2 - 2k \cdot p]} \stackrel{m \ll \Lambda}{\approx} \sum_{i=0}^{\infty} \int_0^\infty [dk] \frac{(m^2)^i}{[k^2]^{i+1}} = 0. \quad (44)$$

We can now solve the integrals in (43) over the whole range and obtain

$$I = \frac{1}{\epsilon} + 1 + \log \frac{\mu^2}{m_X^2} + \frac{m^2}{2m_X^2} \left( 1 + 2 \log \frac{m^2}{m_X^2} \right). \quad (45)$$

This agrees with what we would obtain had we expanded the original integral after integration.  $\square$

**Observation 21.** This is directly connected to the concept of effective field theories (EFT) where we add new operators  $\mathcal{Q}$  and Wilson coefficients  $\mathcal{C}$  to our Lagrangian

$$\mathcal{L} \rightarrow \mathcal{L} + \sum_i \mathcal{C}_i \mathcal{Q}_i. \quad (46)$$

The hard region corresponds to the renormalisation of the Wilson coefficients while the soft region is the loop calculation in the EFT. In this language, the different orders in the expansion correspond directly to the dimensionality of the operators.

This type of expansion is very useful if we want to integrate out a heavy scale. However, we need more tools if we want to deal with light scales as well.

**Definition 22** (Light-cone coordinates). Let  $p_i = E_i(1, \vec{n}_i \beta_i)$  be a high energetic particle ( $\beta = 1 - \mathcal{O}(\lambda^2)$ ), moving in the  $\vec{n}_i$  direction. We now decompose its momentum into light-cone basis vectors  $n_i = (1, \vec{n}_i)/\sqrt{2}$  and  $\bar{n}_i = (1, -\vec{n}_i)/\sqrt{2}$ .<sup>1</sup> We now write any momentum  $p_j$  as

$$p_j = (n_i \cdot p_j) \bar{n}_i + (\bar{n}_i \cdot p_j) n_i + p_j^{(\perp, i)} = p_j^{(+, i)} + p_j^{(-, i)} + p_j^{(\perp, i)} = (n_i \cdot p_j, \bar{n}_i \cdot p_j, p_{j, \perp})_i. \quad (47)$$

For the case  $i = j$  our original energetic particle  $p_i$  is now

$$p_i = (E_i(1 - \beta_i)/\sqrt{2}, E_i(1 + \beta_i)/\sqrt{2}, p_{i, \perp})_i \sim (\lambda^2, 1, \lambda)_i. \quad (48)$$

We have especially  $p_i^2 = m^2 \sim \lambda^2$  as planned

$$p_i^2 = 2p_i^{(+, i)} \cdot p_i^{(-, i)} + (p_i^{(\perp, i)})^2 = m^2 \sim \lambda^2. \quad (49)$$

Here we have used that, since  $\bar{n}^2 = n^2 = n \cdot p_{\perp} = 0$ ,

$$p_j^{(\pm, i)} \cdot p_k^{(\pm, i)} = p_j^{(\pm, i)} \cdot p_k^{(\perp, i)} = 0 \quad (50)$$

**Definition 23** (Momentum regions). Using light-cone coordinates, a region of the loop momentum  $k$  is then defined as a specific choice of parameters  $a$ ,  $b$ , and  $c$  where  $k \sim (\lambda^a, \lambda^b, \lambda^c)_i$  for a given direction  $n_i$ . At this point we expect an infinite number of regions corresponding to the infinite possible choices of  $a$ ,  $b$ , and  $c$ .<sup>2</sup> Fortunately, almost all of the infinite number of regions turn out to be zero.

Common momentum regions that contribute are

$$\text{hard: } k \sim (1, 1, 1) \quad (51a)$$

$$\text{soft: } k \sim (\lambda, \lambda, \lambda) \quad (51b)$$

$$\text{collinear: } k \sim (\lambda^2, 1, \lambda) \quad (51c)$$

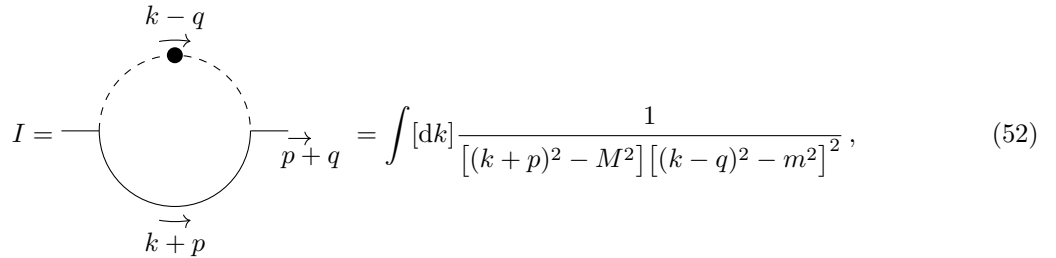
$$\text{ultrasoft: } k \sim (\lambda^2, \lambda^2, \lambda^2), \quad (51d)$$

Note that these scalings are only defined for a given direction. If we have multiple directions of energetic particles as is often the case, we have to distinguish them, leading to more regions.

We might be able to reduce the number of directions if some particles are back-to-back as is the case in  $\gamma^* \rightarrow e^+(n_1)e^-(n_2)$ . The  $(\lambda^2, 1, \lambda)_2$  region is actually the same as the  $(1, \lambda^2, \lambda)_1$  region, reducing the amount of bookkeeping required.

In what follows, I will drop the index for the direction when it is unambiguous.

**Observation 24** (Light particles). Consider the following integral



$$I = \int [dk] \frac{1}{[(k+p)^2 - M^2][(k-q)^2 - m^2]^2}, \quad (52)$$

with  $q^2 = m^2 \ll p^2 = M^2 \sim (p+q)^2 = s$ . A convenient choice is  $p \sim (1, 1, 0)$ ,  $q \sim (0, 1, \lambda)$  which results in

$$p^2 = 2p_+ \cdot p_- = M^2 \sim 1 \quad \text{and} \quad q^2 = q_{\perp}^2 = m^2 \sim \lambda^2. \quad (53)$$

By trying all sensible regions, we find that all but two are scaleless. Consider eg.  $k \sim (\lambda, \lambda, \lambda)$

$$I_s = \frac{1}{\lambda^3} \int [dk] \frac{1}{[2k_+ \cdot q_-]^2 [2k \cdot p]} = 0, \quad (54)$$

where we have used that  $k_+ \cdot q_- = k \cdot q_-$ . The only scaleful regions are  $k \sim (1, 1, 1)$  (hard) and  $k \sim (\lambda^2, 1, \lambda)$  (collinear).

<sup>1</sup>Our definition of  $n$  and  $\bar{n}$  differs from the standard convention by the normalisation factor  $1/\sqrt{2}$ .

<sup>2</sup>One can show that  $c = (a+b)/2$  with other choices resulting in scaleless integrals.

- In the hard region we have  $k \sim (1, 1, 1)$  and we can write

$$I_h = \int [dk] \frac{1}{[(k+p)^2 - M^2][(k-q_-)^2]} + \mathcal{O}(\lambda^2). \quad (55)$$

The leading term is just the original integral with  $m = 0$  since  $q_-^2 = 0$ . In real-world applications, we might be able to obtain this from already known QCD results. We have

$$\begin{aligned} I_h &= \Gamma(1-\epsilon)\Gamma(\epsilon) \int dx_1 dx_2 \delta(\dots) x_1 \frac{(x_1+x_2)^{-2+2\epsilon}}{(M^2 x_2^2 + (M^2-s)x_1 x_2)^\epsilon} \\ &= \Gamma(1-\epsilon)\Gamma(\epsilon) \int_0^1 dx_2 (1-x_2) x_2^{-\epsilon} (M^2 - s + s x_2)^{-\epsilon} \\ &= \frac{(1-x)^\epsilon}{M^2} \left( \frac{M^2}{\mu^2} \right)^{-\epsilon} \left( \frac{\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{1-\epsilon} {}_2F_1 \left[ \begin{matrix} 1-\epsilon, 1+\epsilon \\ 2-\epsilon \end{matrix}; x \right] \right. \\ &\quad \left. + \Gamma(1-\epsilon)\Gamma(\epsilon) {}_2F_1 \left[ \begin{matrix} -\epsilon, 1+\epsilon \\ 2-\epsilon \end{matrix}; x \right] \right) \\ &= \frac{1-x}{M^2} \left( \frac{1}{\epsilon} + \left( 2 + \frac{1}{x} \right) \log(1-x) + \log \frac{\mu^2}{M^2} \right) + \mathcal{O}(m^2). \end{aligned} \quad (56)$$

where we have set  $\delta(1-x_1+x_2)$  and defined  $x = s/(s-M^2)$ . Here we have used a  ${}_2F_1$  function that we will encounter again in Section 4 and expanded it with `HypExp` [15].

- In the collinear region, we have account for a factor  $\lambda^4$  from  $d^4k$ . Hence, we have

$$\begin{aligned} I_c &= \int [dk] \frac{1}{[2k_- \cdot p_+] [k^2 - 2k \cdot q]^2} \\ &= -\Gamma(1-\epsilon)\Gamma(1+\epsilon) \int dx_1 dx_2 \delta(\dots) \frac{x_1^{-1+\epsilon}}{(m^2 x_1 + (m^2 + M^2 - s)x_2)^{1+\epsilon}} \\ &= -M^{-2} \left( \frac{m^2}{\mu^2} \right)^{-2\epsilon} \Gamma(1-\epsilon)\Gamma(\epsilon)(1-x) \\ &= \frac{1-x}{M^2} \left( -\frac{1}{\epsilon} + \log \frac{m^2}{\mu^2} \right). \end{aligned} \quad (57)$$

Here, we once again see that we have lost some complexity since the  $\mathcal{U}$  polynomial is trivial.

Both  $I_c$  and  $I_h$  are divergent but their sum is finite as was the original integral. However,  $I_h \propto (M^2/\mu^2)^{-\epsilon}$  and  $I_c \propto (m^2/\mu^2)^{-\epsilon}$  so that  $\log(M^2/m^2)$  remains

$$I_c + I_h = -\frac{1-x}{xM^2} \left( x \log \frac{M^2}{m^2} + (1-2x) \log(1-x) \right). \quad (58)$$

This is a common feature in the method of regions: all regions are separately more divergent than the total result which can introduce new logarithms.

### 3.2 Parameter space

While the method of region is doubtlessly an invaluable tool, especially when we want to connect its regions to effective theories. However, if all we want to do is calculate integrals without having to bother with a physical intuition, it can be unwieldy, especially since some regions are not visible for a given momentum routing. It turns out we can formulate the method of regions at the level of the Feynman parameters which is much more general and allows for easier automation [16, 17].

Since an important aspect of the method of region is scalelessness, we need to formalise this at the level of the Feynman parameters.

**Lemma 25** (Scaleless integrals). We call an integral scaleless if rescaling the loop momentum and/or the external momenta results in a global factor. These integrals vanish in dimensional regularisation. For example,

$$I = \int [dk] \frac{1}{[k^2]^n} \rightarrow \int [d(\alpha k)] \frac{1}{[(\alpha k)^2]^n} = \alpha^{d-2n} I = 0. \quad (59)$$

In the language of Feynman parameters, the integral vanishes if we can rescale a strict subset of parameters

$$\begin{aligned} \mathcal{U}(x_1, \dots, \alpha x_j, \dots, x_n) &= \alpha^u \mathcal{U}(x_1, \dots, x_j, \dots, x_n) \\ \mathcal{F}(x_1, \dots, \alpha x_j, \dots, x_n) &= \alpha^f \mathcal{F}(x_1, \dots, x_j, \dots, x_n), \end{aligned} \quad (60)$$

for some scaling dimensions  $u$  and  $f$ . For simplicity we only consider  $\mathcal{UF} \rightarrow \alpha^{u+f} \mathcal{UF}$ .

**Theorem 26** (The method of regions in parameter space). To expand a parameter integral, we multiply the small kinematic invariants with appropriate power of  $\lambda$  in  $\mathcal{F}$ . A region is defined as a  $(n+1)$ -dimensional vector  $\vec{v} = (1, v_1, \dots, v_n)$  where  $x_i \sim \lambda^{v_i}$ . We can relate  $\vec{v}$  to the complex hull of the point cloud that is spanned by  $\mathcal{U}$  and  $\mathcal{F}$  (see proof and Observation 27 for details). This is implemented in public codes such as `asy` [16, 17].

**Observation 27** (Example in parameter space). For the example (52) we have  $\vec{v}_h = (1, 0, 0)$  and  $\vec{v}_c = (1, 0, -1)$ . We have hence

$$I_h = -\Gamma(1-\epsilon)\Gamma(1+\epsilon) \int dx_1 dx_2 \delta(\dots) x_2 \frac{(x_1+x_2)^{-1+2\epsilon}}{(M^2 x_1^2 + (M^2-s)x_1 x_2)^{1+\epsilon}}, \quad (61)$$

$$I_c = -\Gamma(1-\epsilon)\Gamma(1+\epsilon) \int dx_1 dx_2 \delta(\dots) \frac{x_2^{2\epsilon}}{(m^2 x_2^2 + (M^2-s)x_1 x_2)^{1+\epsilon}}. \quad (62)$$

The integration of the Feynman parameters is now trivial again as above.

*Proof of Theorem 26.* Consider a term in  $\mathcal{F}$  or  $\mathcal{U}$ , it will have the structure

$$\lambda^{r_0} x_1^{r_1} \dots x_n^{r_n} \equiv \vec{r} = (r_0, r_1, \dots, r_n). \quad (63)$$

We view the terms as a point cloud  $C = C_{\mathcal{F}} \cup C_{\mathcal{U}}$  in an  $(n+1)$  dimensional space.  $\mathcal{F}$  is homogeneous, i.e.  $r_1 + \dots + r_n = \ell + 1$ , meaning that all terms in  $\mathcal{F}$  live on an  $n$  dimensional hyperplane. Were the dimensionality lower than  $n$ , we could factor out a Feynman parameter, making the integral scaleless.

For a given region, this term will scale as

$$\lambda^{r_0} (x_1 \lambda^{v_1})^{r_1} \dots (x_n \lambda^{v_n})^{r_n} \sim \lambda^{r_0 + v_1 r_1 + \dots + v_n r_n} = \lambda^{\vec{v} \cdot \vec{r}}. \quad (64)$$

After expanding  $\mathcal{F}$  and  $\mathcal{U}$  in  $\lambda$ , all remaining terms have to have the same scaling since otherwise the expansion would not have been complete. In other words, the remaining terms belong to a hyperplane orthogonal to  $\vec{v}$ . Points that are above this hyperplane are more suppressed in  $\lambda$  since they have larger  $r_0$ , the power of  $\lambda$ .

If the dimensionality of this hyperplane is lower than  $n$ , the integral is scaleless. Thus we are looking for facets of the envelope of the point cloud  $C$ . The  $\vec{v}$  are the (inwards facing) normal vectors of these facets with  $v_1 > 0$ . This automatically selects only the bottom facets. □

**Observation 28** (Construction of  $\vec{v}$ ). For the example (52) we have

$$\begin{aligned} \mathcal{F} &= M^2 x_1^2 + M^2 x_1 x_2 - s x_1 x_2 + m^2 x_2^2 \lambda, \\ \mathcal{U} &= x_1 + x_2. \end{aligned} \quad (65)$$

This results in the point cloud

$$\begin{aligned} C_{\mathcal{F}} &= \{(0, 2, 0), (0, 1, 1), (0, 1, 1), (1, 0, 2)\} \\ C_{\mathcal{U}} &= \{(0, 1, 0), (0, 0, 1)\} \end{aligned} \quad (66)$$

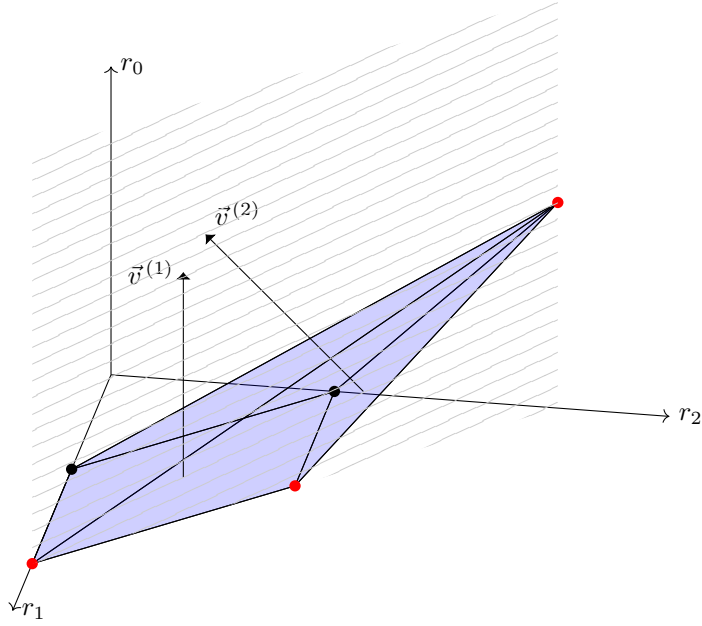


Figure 8: The point clouds  $C_{\mathcal{F}}$  in red and  $C_{\mathcal{U}}$  in black and their convex hull in blue. The grey shaded region corresponds to the requirement that  $\mathcal{F}$  is homogeneous, i.e.  $r_1 + r_2 = \ell + 1$ . An interactive version of this plot can be found on the [course website](#).

as shown in Figure 8. Next, we construct the convex hull of  $C = C_{\mathcal{F}} \cup C_{\mathcal{U}}$ . It has five facets of which we only care about two, those spanned by

$$F_1 = \begin{pmatrix} \vec{r}^{(1)} \\ \vec{r}^{(2)} \\ \vec{r}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} \vec{r}^{(1)} \\ \vec{r}^{(2)} \\ \vec{r}^{(3)} \\ \vec{r}^{(4)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}. \quad (67)$$

We can find the normal vectors  $\vec{v}^{(1)}$  and  $\vec{v}^{(2)}$  by finding the kernel of these matrices as

$$\vec{v}^{(1)} = (1, 0, -1) \quad \text{and} \quad \vec{v}^{(2)} = (1, 0, 0) \quad (68)$$

Here,  $\vec{v}^{(2)}$  corresponds to the hard region and  $\vec{v}^{(1)}$  to the collinear region. We do not, for example, care about this facet

$$F_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \quad (69)$$

which has  $\vec{v}^{(3)} = (0, -1, -1)$ .

## 4 Mellin Barnes integration

When calculating loop integrals, we are often faced with having to integrate polynomials to some non-integer power. Since we can (essentially) only integrate using the definition of the  $\Gamma$  function<sup>3</sup> the most complicated trivially solvable integral is

$$\int_0^\infty dx x^\alpha (a + bx)^\beta = \frac{\Gamma(1 + \alpha)\Gamma(-1 - \alpha - \beta)}{\Gamma(-\beta)} \frac{a^{1+\alpha+\beta}}{b^{1+\alpha}}. \quad (8)$$

It goes without saying that most real-world integrals are more complicated than this.

<sup>3</sup>Technically, we can also solve integrals of the form  $\int dx x^\alpha (1+x)^b (z+x)^c$  using the methods presented in this section. Hence this integral is just a shortcut to what we show here.



## 4.1 The Mellin Barnes theorem

**Observation 29** (Heavy-quark form factor). Consider once again the following one-loop integral

$$I(a) = \int [dk] \frac{1}{[k^2]^a [k^2 + 2k \cdot p] [k^2 - 2k \cdot q]} \quad (70)$$

with  $p^2 = q^2 = m^2$ ,  $(p + q)^2 = s$  and some integer  $a$ . After Feynman parametrisation we find (see eg. Observation 7)

$$I(a) = (-1)^a \frac{\Gamma(1 - \epsilon)\Gamma(a + \epsilon)}{\Gamma(a)} \int dx_1 dx_2 dx_3 \delta(\dots) x_1^{a-1} \frac{(x_1 + x_2 + x_3)^{-2+a+2\epsilon}}{(m^2 x_2^2 + (2m^2 - s)x_2 x_3 + m^2 x_3^2)^{a+\epsilon}}. \quad (71)$$

We can trivially solve the  $x_1$  integral using (8)

$$I(a) = (-1)^a \frac{\Gamma(1 - \epsilon)\Gamma(2 - 2a - 2\epsilon)\Gamma(a + \epsilon)}{\Gamma(2 - a - 2\epsilon)} \int dx_2 dx_3 \delta(\dots) \frac{(x_2 + x_3)^{-2+2a+2\epsilon}}{(m^2 (x_2 + x_3)^2 - s x_2 x_3)^{a+\epsilon}}, \quad (72)$$

but now are stuck. Obviously we cannot just use (8) to solve this integral. However, if we could somehow split the  $\mathcal{F}$  polynomial into the terms  $\propto m^2$  and those  $\propto s$ , we might stand a chance. To do this, we note the Mellin-Barnes theorem.

**Theorem 30** (Mellin-Barnes split). We can split an arbitrary polynomial into two factors at the cost of an integration. You can think of this akin to an inverse Feynman parametrisation

$$\frac{1}{(A + B)^\lambda} = \frac{1}{2\pi i} \frac{1}{\Gamma(\lambda)} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda + z)\Gamma(-z)A^z B^{-z-\lambda}. \quad (73)$$

We ignore the factor  $1/(2\pi i)$  as it will cancel with the one from the residue theorem when we eventually calculate the contour integral. The contour in question is chosen to separate the poles from the left and right  $\Gamma$  functions. We will soon see how this works in practise.

Crucially, we are not allowed to write eg.

$$\frac{\Gamma(1 + z)\Gamma(-z)}{\Gamma(z)} = -\Gamma(1 - z) \quad (74)$$

since it would confuse left and right poles.

**Definition 31** (Left and right  $\Gamma$  functions).  $\Gamma(z)$  functions has infinitely many poles at  $z = -n$ . We hence call  $\Gamma$  functions of the form  $\Gamma(\dots - z)$  right  $\Gamma$  functions as their poles go right towards  $+\infty$ . Similarly,  $\Gamma(\dots + z)$  are left  $\Gamma$  functions.

**Observation 32** (Heavy-quark form factor, Pt. 2). Using (73), we write  $I$  as

$$I(a) = (-1)^a (m^2)^{-a-\epsilon} \frac{\Gamma(1 - \epsilon)\Gamma(2 - 2a - 2\epsilon)}{\Gamma(2 - a - 2\epsilon)} \int_{-i\infty}^{i\infty} dz \left(\frac{-s}{m^2}\right)^z \Gamma(-z)\Gamma(a + \epsilon + z) \int dx_2 dx_3 \delta(\dots) \frac{x_2^z x_3^z}{(x_2 + x_3)^{2+2z}}. \quad (75)$$

Now we can use (8) to solve this integral, using for example  $\delta(\dots) = \delta(1 - x_3)$

$$\int dx_2 dx_3 \delta(\dots) \frac{x_2^z x_3^z}{(x_2 + x_3)^{2+2z}} = \frac{\Gamma(1 + z)^2}{\Gamma(2 + 2z)}. \quad (76)$$

Therefore for  $I$

$$I(a) = (-1)^a (m^2)^{-a-\epsilon} \frac{\Gamma(1 - \epsilon)\Gamma(2 - 2a - 2\epsilon)}{\Gamma(2 - a - 2\epsilon)} \int_{-i\infty}^{i\infty} dz \left(\frac{-s}{m^2}\right)^z \frac{\Gamma(-z)\Gamma(a + \epsilon + z)\Gamma(1 + z)^2}{\Gamma(2 + 2z)}. \quad (77)$$

```
GammaResidue[expr_, {z_, pole_}] := Assuming [
  Element[n, Integers] && n >= 0,
  Residue[expr, {z, pole}]
] /. n! -> Gamma[n + 1]
```

Listing 9: Mathematica calculation of the residue

## 4.2 Solving Mellin Barnes integrals using the residue theorem

**Theorem 33** (Residue theorem for MB). Assuming the MB integrand falls quickly enough, i.e. the arc  $C'$  vanishes (see eg. Figure 10), we can solve a single MB integral using the residue theorem. Consider

$$I = \int_{-i\infty}^{i\infty} dz x^z \underbrace{\left( \prod_{i=1}^{n_L} \Gamma(a_i + z)^{c_i} \right) \left( \prod_{i=1}^{n_R} \Gamma(b_i - z)^{d_i} \right) (\dots)^{-1}}_{f(z)} \quad (78)$$

with  $c_i, d_i > 0$ . Then

$$I = \sum_{i=1}^{n_L} \sum_{n=0}^{\infty} \operatorname{res}_{-a-n} f(z) = \sum_{i=1}^{n_R} \sum_{n=0}^{\infty} \operatorname{res}_{b+n} f(z). \quad (79)$$

**Lemma 34** (Residue formulas). We remember that if  $f(z)$  has a pole of order  $n$  at  $c$ ,

$$\operatorname{res}_c f(z) = \frac{1}{(n-1)!} \lim_{z \rightarrow c} \frac{d^{n-1}}{dz^{n-1}} \left[ (z-c)^n f(z) \right]. \quad (80)$$

If  $f(z) = g(z)h(z)$  and  $h$  is free of poles at  $c$ , this means

$$\operatorname{res}_c f(z) = h(c) \operatorname{res}_c g(z). \quad (81)$$

Finally,

$$\operatorname{res}_{-n} \Gamma(z) = \frac{(-1)^n}{n!}, \quad (82)$$

$$\operatorname{res}_{-n} \Gamma(z)^2 = \frac{2}{(n!)^2} \psi(n+1), \quad (83)$$

$$\operatorname{res}_{-n} \Gamma(z)^3 = \frac{(-1)^n}{2(n!)^4} \left[ \pi^2 + 9\psi(n+1)^2 - 3\psi'(n+1) \right], \quad (84)$$

$$\operatorname{res}_{-n} \psi(z) = -1. \quad (85)$$

Here we have also included the derivative of the  $\Gamma$  function, the digamma function  $\psi(x) = \Gamma'(x)/\Gamma(x)$ . Note that these rules can be implemented in Mathematica with an **Assumption** as shown in Listing 9.

**Observation 35** (Heavy-quark form factor, Pt. 3). For simplicity we close the contour to the right as shown in Figure 10 for  $a = 1$ . We have one residue to calculate since  $n_R = 1$

$$\begin{aligned} I(a) &= (-1)^a (m^2)^{-a-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-2a-2\epsilon)}{\Gamma(2-a-2\epsilon)} \sum_{n=0}^{\infty} \operatorname{res}_n \left[ \left( \frac{-s}{m^2} \right)^z \frac{\Gamma(-z)\Gamma(a+\epsilon+z)\Gamma(1+z)^2}{\Gamma(2+2z)} \right] \\ &= (-1)^{a+1} (m^2)^{-a-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-2a-2\epsilon)}{\Gamma(2-a-2\epsilon)} \sum_{n=0}^{\infty} \left( \frac{s}{m^2} \right)^n \frac{\Gamma(1+n)\Gamma(a+n+\epsilon)}{\Gamma(2+2n)}. \end{aligned} \quad (86)$$

We can solve the remaining sum and find

$$I(a) = (-1)^{a+1} (m^2)^{-a-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-2a-2\epsilon)}{\Gamma(2-a-2\epsilon)} \Gamma(1+\epsilon) {}_2F_1 \left[ 1, \frac{1+\epsilon}{2}; \frac{s}{4m^2} \right]. \quad (87)$$

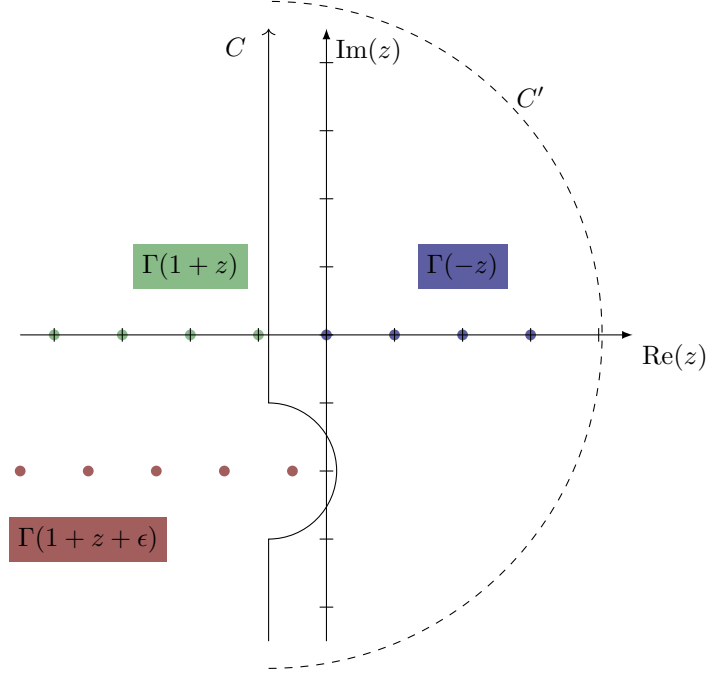


Figure 10: The contour for the MB integral (77) for  $a = 1$

**Definition 36** (Hypergeometric function). We define the hypergeometric function as

$${}_pF_q \left[ \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \quad (88)$$

with the Pochhammer symbol  $a_n = \Gamma(a+n)/\Gamma(a)$  and the factorial  $n! = \Gamma(1+n)$ . For certain values of  $a_i$  and  $b_i$ , these functions can be expanded in  $\epsilon$  using HypExp [15].

### 4.3 Solving Mellin Barnes integrals using Barnes lemma

**Observation 37** (Resolving singularities). Let us consider the special case of  $a = 0$  (ignoring the fact that there are easier ways to obtain this result)

$$I(0) = (m^2)^{-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-2\epsilon)}{\Gamma(2-2\epsilon)} \int_{-i\infty}^{i\infty} dz \left(\frac{-s}{m^2}\right)^z \frac{\Gamma(-z)\Gamma(\epsilon+z)\Gamma(1+z)^2}{\Gamma(2+2z)}. \quad (89)$$

Setting  $\epsilon \rightarrow 0$  is obviously not possible since the left pole from  $\Gamma(\epsilon+z)$  would collide with the right pole from  $\Gamma(-z)$ , meaning the contour would have to cross a pole. However, if we moved the contour as shown in Figure 11 then we can set  $\epsilon \rightarrow 0$ . By crossing a pole with the contour we have to explicitly add the residue of the pole

$$I(0) = (m^2)^{-\epsilon} \Gamma(1-\epsilon) \left[ \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz \left(\frac{-s}{m^2}\right)^z \frac{\Gamma(-z)\Gamma(\epsilon+z)\Gamma(1+z)^2}{\Gamma(2+2z)} - \underbrace{\text{res}_0 [\dots]}_{-\Gamma(\epsilon)} \right] \quad (90)$$

$$= \frac{1}{\epsilon} - \log m^2 + \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz \left(\frac{-s}{m^2}\right)^z \frac{\Gamma(-z)\Gamma(+z)\Gamma(1+z)^2}{\Gamma(2+2z)} + \mathcal{O}(\epsilon). \quad (91)$$

At this stage we may expand in  $\epsilon$  to whatever order we require.

Note that we can also numerically integrate (91).

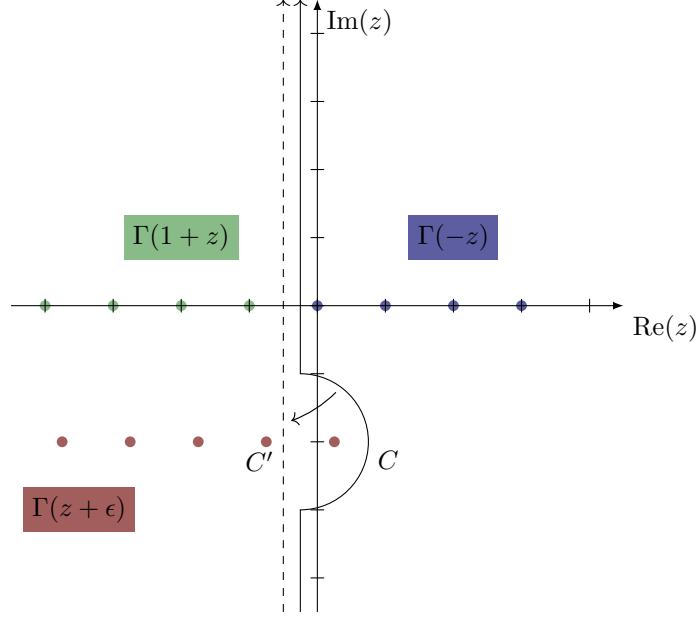


Figure 11: The contour for the MB integral (77) for  $a = 0$

**Observation 38** (Slightly contrived box). Consider this (slightly contrived) example

$$I(a) = \int [dk] \frac{1}{[k^2] [(k-p_1)^2] [(k+p_2)^2] [(k-p_1+p_4)^2]}, \quad (92)$$

with  $p_1^2 = p_3^2 = (p_1 + p_2)^2 = (p_1 - p_3)^2 = s$  and  $p_2^2 = p_4^2 = 0$ . This example could for example occur when calculating the boundary condition for some differential equations of Feynman integrals. There have usually much simpler and often unphysical kinematics and are then evolved to the physics situation. We have

$$I = (-s)^{-2-\epsilon} \Gamma(1-\epsilon) \Gamma(2+\epsilon) \int dx_1 dx_2 dx_3 dx_4 \delta(\dots) \frac{(x_1 + x_2 + x_3 + x_4)^{2\epsilon}}{(x_1 x_2 + x_3 (x_2 + x_4))^{2+\epsilon}}. \quad (93)$$

We split the  $x_3$  term using Mellin Barnes

$$I = (-s)^{-2-\epsilon} \Gamma(1-\epsilon) \int_{-i\infty}^{+i\infty} dz \Gamma(-z) \Gamma(2+\epsilon+z) \int dx_1 dx_2 dx_3 dx_4 \delta(\dots) x_1^{-2-z-\epsilon} x_2^{-2-z-\epsilon} x_3^z (x_3 + x_4)^z (x_1 + x_2 + x_3 + x_4)^{2\epsilon}. \quad (94)$$

We can now obtain

$$I = (-s)^{-2-\epsilon} \frac{\Gamma(1-\epsilon) \Gamma(-\epsilon)}{\Gamma(-2\epsilon)} \int_{-i\infty}^{+i\infty} dz \frac{\Gamma(-z) \Gamma(1+z) \Gamma(-1-\epsilon-z)^2 \Gamma(2+\epsilon+z)}{\Gamma(-\epsilon-z)}. \quad (95)$$

We could obviously solve this with the residue theorem but we have two left poles or two right poles, one of them involving  $\Gamma(\dots - z)^2$ . Let us try to expand in  $\epsilon$  by resolving the singularities

$$\begin{aligned} I &= (-s)^{-2-\epsilon} \frac{\Gamma(1-\epsilon) \Gamma(-\epsilon)}{\Gamma(-2\epsilon)} \left( \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz \frac{\Gamma(-z) \Gamma(1+z) \Gamma(-1-\epsilon-z)^2 \Gamma(2+\epsilon+z)}{\Gamma(-\epsilon-z)} - \text{res}_{-1-\epsilon}(\dots) \right) \\ &= 2(-s)^{-2-\epsilon} \left[ \frac{1}{\epsilon^2} - 2\zeta_2 + \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz \Gamma(1+z) \Gamma(2+z) \Gamma(-1-z)^2 \right. \\ &\quad \left. - \left( 2\zeta_3 + \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz \Gamma(1+z) \Gamma(2+z) \Gamma(-1-z)^2 (\psi(-z) - 2\psi(-1-z) + \psi(2+z)) \right) \epsilon \right]. \quad (96) \end{aligned}$$

**Theorem 39** (Barnes Lemma). We have two Barnes Lemma that can be used to calculate integrals

$$\int_{-i\infty}^{+i\infty} dz \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z) = \frac{\Gamma(\lambda_{13})\Gamma(\lambda_{14})\Gamma(\lambda_{23})\Gamma(\lambda_{24})}{\Gamma(\lambda_{1234})}, \quad (97)$$

$$\int_{-i\infty}^{+i\infty} dz \frac{\Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 + z)\Gamma(\lambda_4 - z)\Gamma(\lambda_5 - z)}{\Gamma(\lambda_{12345} + z)} = \frac{\Gamma(\lambda_{14})\Gamma(\lambda_{24})\Gamma(\lambda_{34})\Gamma(\lambda_{15})\Gamma(\lambda_{25})\Gamma(\lambda_{35})}{\Gamma(\lambda_{1245})\Gamma(\lambda_{1345})\Gamma(\lambda_{2345})}, \quad (98)$$

with  $\lambda_{12} = \lambda_1 + \lambda_2$  etc. The Barnes Lemma are particularly useful when dealing with multiple MB integrals. Further, they can be used to generate further identities. For example, taking the derivative of (97) w.r.t.  $\lambda_1$

$$\int_{-i\infty}^{+i\infty} dz \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z)\psi(\lambda_1 + z) = \frac{\Gamma(\lambda_{13})\Gamma(\lambda_{14})\Gamma(\lambda_{23})\Gamma(\lambda_{24})}{\Gamma(\lambda_{1234})} \times \left[ \psi(\lambda_{13}) + \psi(\lambda_{14}) - \psi(\lambda_{1234}) \right]. \quad (99)$$

**Observation 40** (Slightly contrived box, Pt. 2). We can now calculate the  $\mathcal{O}(\epsilon^0)$  term of (96), and by taking the derivative w.r.t. the  $\lambda_i$

$$I = 2(-s)^{-2-\epsilon} \left[ \frac{1}{\epsilon^2} - \zeta_2 - \left( -\zeta_3 + \gamma_E \zeta_2 + \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz \Gamma(1+z)\Gamma(2+z)\Gamma(-1-z)^2 \psi(-z) \right) \epsilon \right]. \quad (100)$$

It is possible to write  $\psi(-z)$  as  $\psi(1+z)$  and again use the Barnes Lemmas. Alternatively, we can note that

$$\Gamma(-1-z)\Gamma(2+z) = -\frac{1}{1+z}\Gamma(-z)\Gamma(2+z) = -\Gamma(1+z)\Gamma(-z). \quad (101)$$

We are now allowed do this, since we have specified the contour explicitly and no longer need to distinguish left and right poles. We write

$$\begin{aligned} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz \Gamma(1+z)\Gamma(2+z)\Gamma(-1-z)^2 \psi(-z) &= - \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz \Gamma(1+z)^2 \Gamma(-z)\Gamma(-1-z)\psi(-z) \\ &= -2\zeta_3 - \gamma_E \zeta_2. \end{aligned} \quad (102)$$

With this

$$I = 2(-s)^{-2-\epsilon} \left[ \frac{1}{\epsilon^2} - \zeta_2 - 3\zeta_3 \epsilon \right]. \quad (103)$$

Naturally, doing this manually quickly becomes annoying, the higher we want to go in  $\epsilon$ .

**Observation 41** (Strategies). In the example above, the integral has trivial dependence on external kinematics. In general, this means that it can be written as  $I = (Q^2)^{\ell\epsilon} f(\epsilon)$  where  $f$  is some function that only depends on the dimensionality of spacetime. Using the methods described in Observation 37, we are able to expand  $f$  around  $\epsilon = 0$  and calculate the remaining integrals numerically. We will shortly see a particularly clever way to exploit this.

If that were not the case, we have to explicitly sum the residues. Mathematica is capable of doing this in a limited number of cases and the FORM code `XSummer` [18] covers many more. However, all of this works only if there is only a single integration left. Finding a series expansion for multiple MB integrals is highly non-trivial and subject to active research (eg. [19]).

**Theorem 42** (PSLQ algorithm). Given some real (notionally transcendental) numbers  $a_i$ , we can find rational numbers  $r_i$  such that

$$r_0 a_0 + r_1 a_1 + \dots + r_n a_n = 0 \quad (104)$$

using black magic, also known as number theory. An implementation of this algorithm can be found with these lecture notes.

**Observation 43** (Using PSLQ). If we can guess a basis of transcendental numbers such as

$$\{a_i\}_{i=1,\dots,n} = \{1, \zeta_2, \zeta_3, \zeta_4, \dots, \log 2, \log 2\zeta_2, \dots\}, \quad (105)$$

we can numerically evaluate a MB integral to very high precision using `NIntegrate` as  $a_0$  and find an analytic solution

$$a_0 = \frac{r_1}{r_0} a_1 + \frac{r_2}{r_0} a_2 + \dots + \frac{r_n}{r_0} a_n. \quad (106)$$

Depending on which factors are pulled out before  $a_0$  is calculated, we might need to add  $\gamma_E$  and  $\pi$  to the basis. The larger  $n$  is, the more digits we need in  $a_0$  which quickly gets expensive.

In our example, expanding (96) to  $\epsilon^4$ , we have for the integral part

$$\begin{aligned} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz(\dots) &= 3.2898681336964528729448303332920503784378998024136 \\ &- 2.4041138063191885707994763230228999815299725846810 \epsilon^1 \\ &+ 2.1646464674222763830320073930823358055495019038375 \epsilon^2 \\ &- 9.9830729114759243254510727388545706244427677056444 \epsilon^3 \\ &- 2.8676528331332807519186620879948731784305070422237 \epsilon^4 \end{aligned} \quad (107)$$

calculated to 50 digits accuracy (cf. Listing 12). We can now use PSLQ with the following basis

$$\left\{ \underbrace{\zeta_2}_{w=2}, \underbrace{\zeta_3}_{w=3}, \underbrace{\zeta_4}_{w=4}, \underbrace{\zeta_5, \zeta_2\zeta_3}_{w=5}, \underbrace{\zeta_6, \zeta_3\zeta_3}_{w=6} \right\} \quad (108)$$

to find

$$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dz(\dots) = 2\zeta_2 - 2\zeta_3\epsilon^1 + 2\zeta_4\epsilon^2 - (2\zeta_5 - 4\zeta_2\zeta_3)\epsilon^3 - \left(\frac{17}{2}\zeta_6 + 4\zeta_3^2\right)\epsilon^4 + \mathcal{O}(\epsilon^5). \quad (109)$$

The basis has the added property that it has constant transcendentality.

In [20], Stefano Laporta calculated 1100 digits of the 4-loop QED correction to  $(g-2)_e$  and fitted a basis with more than hundred elements.

**Definition 44** (Transcendentality). For each element of the PSLQ basis  $a$  we can define a transcendentality  $w(a)$ . This property is multiplicative, i.e.  $w(a_1 \times a_2) = w(a_1) + w(a_2)$ . Some transcendentalities can be found in Table 13. Note that even though (108) has what is called constant transcendentality (at each order  $w$  is constant), this is not a very useful concept for QCD. However, certain symmetries such as conformal symmetry result in constant transcendentality at the level of the matrix element.

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```

integrand=Gamma[1-ep]*Gamma[-ep]*Gamma[-1-ep-z]^2*Gamma[-z]*Gamma[1+z]*
  ↪ Gamma[2+ep+z]/Gamma[-2*ep]/Gamma[-ep-z];

integrandC = integrand/.{
  z -> -1/2 + I * v
};

num = Table[
  NIntegrate[
    SeriesCoefficient[
      integrandC,
      {ep,0,i},
    ],
    {v,-Infinity, Infinity},
    WorkingPrecision -> 100,
    PrecisionGoal -> 50,
    AccuracyGoal -> 50
  ],
  {i, 0, 4}
];
MyPSLQ[num, {
  Zeta[2],
  Zeta[3],
  Zeta[4],
  Zeta[5], Zeta[2] Zeta[3],
  Zeta[6], Zeta[3] Zeta[3]
}]

```

Listing 12: Numerical evaluation and PSLQ

$w$	#	values
0	1	1
1	1	$\log 2$
2	2	$\zeta_2, \log^2 2$
3	3	$\zeta_3, \zeta_2 \log 2, \log^3 2$
4	5	$\zeta_4, \zeta_3 \log 2, \zeta_2 \log^2 2, \log^4 2, \text{Li}_4(\frac{1}{2})$
5	8	$\zeta_5, \zeta_2 \zeta_3, \zeta_4 \log 2, \zeta_3 \log^2 2, \zeta_2 \log^3 2, \log^5 2, \log 2 \text{Li}_4(\frac{1}{2}), \text{Li}_5(\frac{1}{2})$
6	13	$\zeta_6, \zeta_3 \zeta_3, \zeta_5 \log 2, \zeta_2 \zeta_3 \log 2, \zeta_4 \log^2 2, \zeta_3 \log^3 2, \zeta_2 \log^4 2, \log^6 2, \zeta_2 \text{Li}_4(\frac{1}{2}), \log^2 2 \text{Li}_4(\frac{1}{2}), \log 2 \text{Li}_5(\frac{1}{2}), \text{Li}_6(\frac{1}{2}) H_{0,0,0,0,1,1}(\frac{1}{2})$
$w$		$\zeta_i, \text{Li}_i(\frac{1}{2}), \dots$

Table 13: Some transcendental numbers up to  $w = 6$ , adapted from [21]

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