Feynman diagrams without QFT

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Reminder: Interaction picture

Let us split the Hamiltonian $H = H_0 + H'$ where H_0 is solvable exactly, i.e. we can find U_0 s.t.

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U_0 = H_0 U_0\,.$$

We now define $U = U_0 U'$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U = \underbrace{\mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}t}U_0\right)}_{H_0U_0} U' + \mathrm{i}U_0\frac{\mathrm{d}}{\mathrm{d}t} \stackrel{!}{=} HU = (H_0+H')U_0U'\,,$$

$$\Rightarrow \qquad \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}U' = U_0^\dagger H' U_0 U' = H_I U'\,, \tag{1}$$

defining the *interaction picture* H_I. We can solve (1) iteratively

$$\begin{aligned} U(\infty,-\infty) &= 1 - \mathrm{i}\int H_I(\tau)\mathrm{d}\tau + (-\mathrm{i})^2\int \mathrm{d}\tau_2\int \mathrm{d}\tau_1 H_I(\tau_2)H_I(\tau_1) + ... \\ &= T\exp\left(-\mathrm{i}\int \mathrm{d}\tau H_I(\tau)\right) \end{aligned}$$

where T ensures the time-ordering, i.e.

$$(-\mathrm{i})^2\int \mathrm{d}\tau_2\int \mathrm{d}\tau_1 H_I(\tau_2)H_I(\tau_1) = \frac{(-\mathrm{i})^2}{2!}T\Big\{H_I(\tau_2)H_I(\tau_1)\Big\}\,.$$

The S matrix in quantum field theory

The object we want to compute in QFT is the S matrix

$$S = U_I(\infty, -\infty) = T \exp\left(-\mathrm{i} \int \mathrm{d}\tau H_I(\tau)\right) = 1 + \mathrm{i}\mathcal{T}\,,$$

where H_I is hopefully small so that the series converges fast enough. Now, computing S in all generality is obviously impossible. Let us instead look at the transition amplitude between some initial state $|i\rangle$ and some final state $|f\rangle$. This is

$$\langle f|S|i\rangle = \langle f|i\rangle - (2\pi)^4 \delta(P_f - P_i)\mathcal{A}_{fi}\,,$$

$\mathcal{A}$ is now the invariant transition amplitude. Squaring it will give us a matrix element[1].

The S matrix is unitary

$$1 = \sum_f |\langle f|S|i\rangle|^2 = \sum_f \left\langle i\middle|S^\dagger\middle|f\right\rangle \langle f|S|i\rangle = \left\langle i\middle|S^\dagger S\middle|i\right\rangle\,.$$

If we split $S = 1 + \mathrm{i}\mathcal{T}$ we can see

$$\begin{aligned} 1 = SS^\dagger &= 1 + \mathrm{i}\mathcal{T} - \mathrm{i}\mathcal{T}^\dagger + \mathcal{T}\mathcal{T}^\dagger \\ \mathcal{T} - \mathcal{T}^\dagger =& 2\Im\mathcal{T} = \mathrm{i}\mathcal{T}\mathcal{T}^\dagger\,. \end{aligned}$$

If we calculate

$$2\langle f|\Im\mathcal{T}|i\rangle = \sum_n \left\langle f\middle|\mathcal{T}^\dagger\middle|n\right\rangle \langle n|\mathcal{T}|i\rangle$$

and set $f = i$, we find the total cross section $i \to$ whatever

$$\sum_n |\langle n|\mathcal{T}|i\rangle|^2 = 2\langle i|\Im\mathcal{T}|i\rangle\,.$$

Now we want to turn this into a relativistic quantum field theory. For this we turn

$$S = T\exp\left(-\mathrm{i}\int \mathrm{d}\tau H_I(\tau)\right) \to T\exp\left(+\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\text{int}}(x)\right)\,,$$

where $\mathcal{L}_{\text{int}}$ is the interaction part of the Lagrangian density.

Let us derive Feynman rules for a toy QFT

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_\phi^2}{2}\phi^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \frac{m_\chi^2}{2}\chi^2 - g\phi\chi^2 = \mathcal{L}_0 + \mathcal{L}_{\text{int}}\,.$$

This theory has two particles ϕ and χ that can interact (later ϕ could be the photon and χ a fermion). We assume that the coupling g is small.

The S matrix is now

$$S = T\exp\left(-\mathrm{i}\int \mathrm{d}^4x\, g\phi\chi^2\right)\,. \tag{2}$$

S is now an operator in the Fock space, just as ϕ and χ! To understand this, let us look at the free theory.

[1]Different sources are not consistent with what is called a matrix element and what an amplitude. Often $\mathcal{A}$ is called $\mathcal{M}$

The free theory

Let us begin with the free theory $\mathcal{L}_0$. The Euler-Lagrange equation for ϕ or χ is

$$\partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}_0}{\partial \phi} = 0\,.$$

This gives us the Klein-Gordon equation

$$(\partial^2 + m_\phi^2)\phi = 0\,. \tag{3}$$

Let us solve this equation

$$\phi(x) = \int [\mathrm{d}k]\Big(a(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\boldsymbol{k})\mathrm{e}^{\mathrm{i}kx}\Big)\,, \tag{4}$$

$$\chi(x) = \int [\mathrm{d}k]\Big(b(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}kx} + b^\dagger(\boldsymbol{k})\mathrm{e}^{\mathrm{i}kx}\Big)\,, \tag{5}$$

where $[dk]$ is the Lorentz invariant phase space measure. The operators $a(\boldsymbol{k})$ ($a^\dagger(\boldsymbol{k})$), when acting on the vacuum $|0\rangle$ destroy (generate) a particle with momentum $\boldsymbol{k}$. This means that $\phi(x)$ is also an operator.

The operators a and b have commutation relations like the harmonic oscillator

$$[a(k), a^\dagger(q)] = (2\pi)^3 2\omega_q \delta(\boldsymbol{q} - \boldsymbol{k}) \equiv \delta(q-k)\,, \tag{6}$$

$$[a(k), a(q)] = [a(k), b^\dagger(q)] = 0\,. \tag{7}$$

The δ function is normalised such that $\int [\mathrm{d}k]\delta(q-k) = 1$.

By defining the canonical momenta

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi$$

We find equal time commutation relations

$$[\phi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = \mathrm{i}\delta(\boldsymbol{x} - \boldsymbol{y})\,.$$

The objects $|i\rangle$ and $\langle f|$ are now states in the Fock space. The state $a^\dagger(\boldsymbol{k})\,|0\rangle$ is the state with one ϕ particle with momentum $\boldsymbol{k}$ and mass m_ϕ. Right now, we do not have internal degrees of freedom like spin.

The easiest thing to calculate is

$$\tilde G(x) = \langle 0|T\{\phi(x)\phi(0)\}|0\rangle\ .$$

This is the probability of a particle, being created at at $x = 0$ and destroyed at x. For simplicity we will assume that $x_0 > 0$

$$\tilde G(x) = \langle 0|\phi(x)\phi(0)|0\rangle = \int [\mathrm{d}k][\mathrm{d}q]\,\langle 0|\,\Big(a_k\mathrm{e}^{-\mathrm{i}kx} + \cancel{a_k^\dagger}\mathrm{e}^{\mathrm{i}kx}\Big)\Big(\cancel{a_q} + a_q^\dagger\Big)\,|0\rangle\ .$$

Using

$$a_k a_q^\dagger\,|0\rangle = [a_k, a_q^\dagger]\,|0\rangle + a_q^\dagger a_k\,|0\rangle = \delta(k-q)\,|0\rangle \tag{8}$$

we write

$$\tilde{G}(x) = \int [\mathrm{d}k][\mathrm{d}q]\mathrm{e}^{-\mathrm{i}kx} \left\langle 0 \middle| a_k a_q^\dagger \middle| 0 \right\rangle = \int [\mathrm{d}k]\mathrm{e}^{-\mathrm{i}kx}$$
$$= \int [\mathrm{d}k]\Big(\theta(x_0)\mathrm{e}^{-\mathrm{i}kx} + \theta(-x_0)\mathrm{e}^{\mathrm{i}kx}\Big)\,.$$

Let us Fourier transform this

$$G(p) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}0^+}\,. \tag{9}$$

This is the *Feynman propagator* and also the Green's function of the Klein-Gordon operator

$$(\partial_\mu \partial^\mu + m^2)\tilde{G}(x) = -\mathrm{i}\delta(x)\,. \tag{10}$$

The interaction

Let us now use what we have learned to calculate $\phi \to \chi\chi$, i.e.

$$|i\rangle = a^\dagger(p_1)\,|0\rangle \qquad \text{and} \qquad |f\rangle = b^\dagger(p_2)b^\dagger(p_3)\,|0\rangle\ .$$

The S matrix element is now

$$\langle i|S|f\rangle = \cancel{\left\langle 0 \middle| b_2 b_3 a_1^\dagger \middle| 0 \right\rangle} + (-\mathrm{i}g)\,\langle 0|\, b_2 b_3 \int \mathrm{d}^4x[\mathrm{d}k_4][\mathrm{d}k_5][\mathrm{d}k_6]\Big(a_4\mathrm{e}^{-\mathrm{i}k_4x} + \cancel{a_4^\dagger \mathrm{e}^{\mathrm{i}k_4x}}\Big)$$
$$\Big(\cancel{b_5\mathrm{e}^{-\mathrm{i}k_5x}} + b_5^\dagger \mathrm{e}^{\mathrm{i}k_5x}\Big)\Big(\cancel{b_6\mathrm{e}^{-\mathrm{i}k_6x}} + b_6^\dagger \mathrm{e}^{\mathrm{i}k_6x}\Big)a_1^\dagger\,|0\rangle\ .$$

Using (8) we see why the b_5 term vanishes

$$\int [\mathrm{d}k_5]\ b_2 b_3 \mathrm{e}^{\cdot\cdot} b_5 b_6^\dagger\,|0\rangle = \int [\mathrm{d}k_5]\ \mathrm{e}^{\cdot\cdot}\delta(k_5 - k_6) b_2 b_3\,|0\rangle = 0\,.$$

We now have and use (8) again and again

$$\langle i|S|f\rangle = (-\mathrm{i}g)\int \mathrm{d}^4x[\mathrm{d}k_4][\mathrm{d}k_5][\mathrm{d}k_6]\mathrm{e}^{\mathrm{i}(k_5+k_6-k_4)x}\,\langle 0|\, b_2 b_3 b_5^\dagger b_6^\dagger a_4 a_1^\dagger\,|0\rangle$$
$$= (-\mathrm{i}g)\int \mathrm{d}^4x[\mathrm{d}k_4][\mathrm{d}k_5][\mathrm{d}k_6]\mathrm{e}^{\mathrm{i}(k_5+k_6-k_4)x}\delta(p_2-k_6)\delta(p_3-k_5)\delta(k_4-p_1)\underbrace{\langle 0|0\rangle}_{1}$$
$$= (-\mathrm{i}g)\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}(p_3+p_2-p_1)x} = (2\pi)^4\delta(p_2+p_3-p_1)\,(-\mathrm{i}g)\,.$$

First note that we have momentum conservation, i.e. $p_1 = p_2 + p_3$. The interesting bit is, however, the $(-\mathrm{i}g)$-part. This is a Feynman rule. This means that for all $\phi\chi\chi$ vertices we have to write $-\mathrm{i}g$. Note that this is also the coefficient of $\mathrm{i}\mathcal{L}_{\text{int}}$, which is how Feynman rules are sometimes derived.

A more involved example: $2 \to 2$ scattering

Let us consider the scattering of $\chi(p_1)\chi^\dagger(p_2) \to \chi(p_3)\chi^\dagger(p_4)$. We have

$$|i\rangle = b_1^\dagger d_2^\dagger |0\rangle \qquad \text{and} \qquad \langle f| = \langle 0| b_3 d_4 \,.$$

We want to calculate

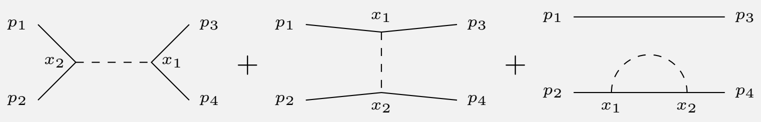
$$T = (-\mathrm{i}g^2)\frac{1}{2!}\langle 0| b_3 d_4 T\left\{ \int \mathrm{d}^4x_1 \underbrace{\phi(x_1)}_{k_5}\underbrace{\chi(x_1)}_{k_6}\underbrace{\chi^\dagger(x_1)}_{k_7}\int \mathrm{d}^4x_2 \underbrace{\phi(x_2)}_{k_8}\underbrace{\chi(x_2)}_{k_9}\underbrace{\chi^\dagger(x_2)}_{k_0}\right\} b_1^\dagger d_2^\dagger |0\rangle \;.$$

We now could substitute the expressions for ϕ and χ and commute all the expressions until we arrive at a result. But we could also not do that and think about what δ functions we will get

$$\begin{aligned}
T &= (-\mathrm{i}g^2)\frac{1}{2!}\langle 0| b_3 d_4 T\left\{ \int \mathrm{d}^4x_1\phi(x_1)\chi(x_1)\chi^\dagger(x_1)\int \mathrm{d}^4x_2\phi(x_2)\chi(x_2)\chi^\dagger(x_2)\right\} b_1^\dagger d_2^\dagger |0\rangle \\
&+ (-\mathrm{i}g^2)\frac{1}{2!}\langle 0| b_3 d_4 T\left\{ \int \mathrm{d}^4x_1\phi(x_1)\chi(x_1)\chi^\dagger(x_1)\int \mathrm{d}^4x_2\phi(x_2)\chi(x_2)\chi^\dagger(x_2)\right\} b_1^\dagger d_2^\dagger |0\rangle \\
&+ (-\mathrm{i}g^2)\frac{1}{2!}\langle 0| b_3 d_4 T\left\{ \int \mathrm{d}^4x_1\phi(x_1)\chi(x_1)\chi^\dagger(x_1)\int \mathrm{d}^4x_2\phi(x_2)\chi(x_2)\chi^\dagger(x_2)\right\} b_1^\dagger d_2^\dagger |0\rangle
\end{aligned}$$

where for example $\chi(x_2)\cdots b_2^\dagger|0\rangle$ indicates that the contraction would give a $\delta(p_1 - k_9)$. We now have two terms corresponding to two Feynman diagrams. Let us build all possible contraction involving external particles

$$\begin{aligned}
T &= (-\mathrm{i}g^2)\frac{1}{2!}\langle 0| b_3 d_4 T\left\{ \int \mathrm{d}^4x_1\phi(x_1)\chi(x_1)\chi^\dagger(x_1)\int \mathrm{d}^4x_2\phi(x_2)\chi(x_2)\chi^\dagger(x_2)\right\} b_1^\dagger d_2^\dagger |0\rangle \\
&+ (-\mathrm{i}g^2)\frac{1}{2!}\langle 0| b_3 d_4 T\left\{ \int \mathrm{d}^4x_1\phi(x_1)\chi(x_1)\chi^\dagger(x_1)\int \mathrm{d}^4x_2\phi(x_2)\chi(x_2)\chi^\dagger(x_2)\right\} b_1^\dagger d_2^\dagger |0\rangle \\
&+ (-\mathrm{i}g^2)\frac{1}{2!}\langle 0| b_3 d_4 T\left\{ \int \mathrm{d}^4x_1\phi(x_1)\chi(x_1)\chi^\dagger(x_1)\int \mathrm{d}^4x_2\phi(x_2)\chi(x_2)\chi^\dagger(x_2)\right\} b_1^\dagger d_2^\dagger |0\rangle
\end{aligned}$$

$= \int \mathrm{d}^4x_1\mathrm{d}^4x_2 \Big($  $+ \cdots \Big)$

All diagrams that are obtained from contracting external particles are *disconnected* and do not contribute to T.

What about the internal contraction? Because ϕ and χ commute, we can consider

$$\langle 0| T\left\{\cdots\phi(x_1)\cdots\phi(x_2)\cdots\right\}|0\rangle = \langle 0| T\{\phi(x_1)\phi(x_2)\}|0\rangle = \tilde{G}(x_1 - x_2)$$

which is just the Feynman propagator. Let us combine what we know

$$\begin{aligned}
T &= (-\mathrm{i}g^2)\frac{1}{2!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \mathrm{e}^{\mathrm{i}(p_4+p_3)x_1}\mathrm{e}^{-\mathrm{i}(p_1+p_2)x_2}\langle 0|\,T\{\phi(x_1)\phi(x_2)\}\,|0\rangle \\
&+ (-\mathrm{i}g^2)\frac{1}{2!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \mathrm{e}^{-\mathrm{i}(p_1-p_3)x_1}\mathrm{e}^{-\mathrm{i}(p_2-p_4)}\langle 0|\,T\{\phi(x_1)\phi(x_2)\}\,|0\rangle \\
&= (-\mathrm{i}g^2)\frac{1}{2!}(2\pi)^4\delta(p_1+p_2-p_3-p_4)\frac{\mathrm{i}}{(p_1+p_2)^2-m^2+\mathrm{i}0^+} \\
&+ (-\mathrm{i}g^2)\frac{1}{2!}(2\pi)^4\delta(p_1+p_2-p_3-p_4)\frac{\mathrm{i}}{(p_1-p_3)^2-m^2+\mathrm{i}0^+}\,.
\end{aligned}$$

By swapping $x_1 \leftrightarrow x_2$ we get an additional factor 2.

$$T = (2\pi)^4\delta(p_1+p_2-p_3-p_4)\times\left((-\mathrm{i}g^2)\frac{\mathrm{i}}{(p_1+p_2)^2-m^2}+(-\mathrm{i}g^2)\frac{\mathrm{i}}{(p_1-p_3)^2-m^2}\right)\,.$$

We can now formuate a general procedure for this theory

1. Draw all connected diagrams up to a certain power in g
2. Attach directed momenta to each line
3. For each $\phi\chi\chi$ vertex, attach $-\mathrm{i}g$
4. Integrate over all unconstrained momenta

QED

We now have all the tools to see how QED works[2]. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\mathrm{i}\not{D} - m)\psi\,.$$

The objects here are as follows

- two (complex) spinor fields ψ and $\bar{\psi} = \psi^\dagger\gamma^0$. These are four-component vectors containing the spin up and down operators for particles and anti-particles.
- We denote $\not{x} = x_\mu\gamma^\mu$. These γ matrices are 4×4 matrices that fulfill a Cliford algebra, i.e.

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{11}$$

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the field strength tensor
- the 4-component vector fields A_μ of Maxwell
- the covariant derivative $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$

Again, we have the free photon and electron fields

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi \qquad \text{and} \qquad \mathcal{L}_{\text{int}} = e\bar{\psi}\not{A}\psi\,.$$

[2] though like all QFT lecture, we will speed the discussion up a lot and will not derive the Feynman rules

The Euler Lagrange equations for $\mathcal{L}_0$ are the (more or less) classical Maxwell equations for A_μ (just keep in mind that A_μ is an operator) and the Dirac equation for ψ.

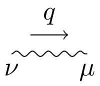
The Dirac equation $(\mathrm{i}\not{\partial} - m)\psi = 0$ has solutions

$$\psi_\alpha = \int [\mathrm{d}k] \sum_{s=1}^{2} \Big(b(s,\boldsymbol{k}) u_\alpha \mathrm{e}^{-\mathrm{i}kx} + d^\dagger(s,\boldsymbol{k}) v_\alpha(s,\boldsymbol{k}) \mathrm{e}^{\mathrm{i}kx} \Big) ,$$

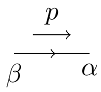
where u (v) describe particles (anti-particles). s indicated the dependence of the spin-state. It is possible to write u and v down explictly using s as four-vectors. We will not be doing this and only note that u and v fulfil the following equation of motion[3]

$$(\not{k} - m)u(s,\boldsymbol{k}) = (\not{k} + m)v(s,\boldsymbol{k}) = 0 .$$

The Green's function give us the propagators

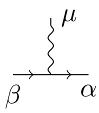


$$= \frac{-\mathrm{i}g_{\mu\nu}}{p^2 + \mathrm{i}0^+} ,$$



$$= \Big[(\not{p} - m)^{-1} \Big]_{\alpha\beta} = \frac{[\not{p} + m]_{\alpha\beta}}{p^2 - m^2 + \mathrm{i}0^+} .$$

We can kind of just read of the $e\bar{e}\gamma$ vertex



$$= -\mathrm{i}e\gamma^\mu_{\beta\alpha} .$$

Finally, we need the asymptotic states. For example an incoming (outgoing) e^- is

$$\psi b^\dagger \left|0\right\rangle \to u \qquad \text{and} \qquad \left\langle 0\right| b\bar{\psi}^\dagger \to \bar{u} .$$

Similarly we use v-type spinors for positrons.

Fermi's golden rule

We now can calculate the amplitude $\mathcal{A}$. However, we want something like a cross-section. For that we note, that the probability this process is (up to normalisation)

$$\left|\mathcal{T}^2\right|^2 = \left|\mathrm{i}(2\pi)^4\delta(P_f - P_i)\mathcal{A}\right|^2$$

Unfortunately, $|\delta|^2$ is meaningless. This is because our amplitude was written in term of plain waves. One way to fix this is to construct wave-packets. We will instead put the system in a box of size $L \to \infty$ and consider a time interval $-T < t < T$. The δ function now becomes

$$(2\pi)^4\delta(P_f - P_i) = I(E_f - E_i, T) I^3(\boldsymbol{P}_f - \boldsymbol{P}_i, L) .$$

The function I should, for $T \to \infty$ approach the δ function. A sensible choice is

$$I(\Delta E, T) = \frac{2}{\Delta E} \sin \frac{\Delta E\, T}{2} ,$$

[3] We will from now on drop the spin index dependence

that features

$$I(\Delta E, T)^2 = 2\pi T \delta(\Delta E)\,.$$

We now have

$$\left|(2\pi)^4\delta(P_f - P_i)\right|^2 \approx L^3 T (2\pi)^4 \delta(P_f - P_i)$$

Because we normalised the fields with $2E$ particles per volume, we divide by $2EV = \int u^\dagger u$ per particle. The transition rate, i.e. probability per unit time, is thus

$$\frac{1}{T}|\mathcal{A}|^2 VT(2\pi)^4\delta(P_f - P_i)\prod_i \frac{1}{2E_i V}\prod_f \frac{1}{2E_f V}$$

where the i (f) product runs over initial (final) particles.

Because the box is finite in size there are only

$$\mathrm{d}n = \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}V \to \prod_f \frac{\mathrm{d}^3\boldsymbol{k}_f V}{(2\pi)^3}$$

states between $\boldsymbol{k}$ and $\boldsymbol{k} + \mathrm{d}\boldsymbol{k}$. This product now only runs over final states. The transition rate into a particular part of phase space is now

$$\mathrm{d}W = |\mathcal{A}|^2 V \prod_{\text{in}} \frac{1}{2E_i V} \underbrace{(2\pi)^4\delta(P_f - P_i)\prod_f \frac{\mathrm{d}^3\boldsymbol{k}_f}{(2\pi)^3 2E_f V}}_{\mathrm{d}\Phi}$$

$\mathrm{d}\Phi$ is now called the Lorentz invariant phase space (which it is despite its looks).

For decay rates and cross section V will cancel. Consider a beam of one particle per V with velocity v. It has a flux of $N_0 = v/V$. Now let us generalise to two beams with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

The cross section is thus

$$\mathrm{d}\sigma = \frac{\mathrm{d}W}{N_0} = \frac{1}{|\boldsymbol{v}_1 - \boldsymbol{v}_2|}\frac{1}{4E_1E_2}|\mathcal{A}|^2\mathrm{d}\Phi\,.$$

Note the vectors $\boldsymbol{v}_i$ are added like vectors and not like relativistic velocities, i.e. $|\boldsymbol{v}_1 - \boldsymbol{v}_2| = 2$ for massless particles. For $2 \to 2$ scattering one can easily show that

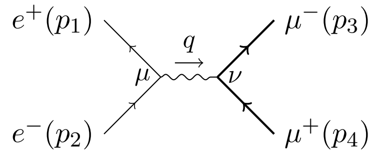
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s}\frac{1}{|\boldsymbol{p}_1|^2}|\mathcal{A}|^2\,,$$

with the Mandelstamm variable t and the centre-of-mass energy s. For massless particles, the t integration goes from $t = 0$ to $t = s$ with $|\boldsymbol{p}_1| = \sqrt{s}/2$.

Muon pair production $e^+e^- \to \mu^+\mu^-$

We are now ready to calculate a matrix element. As an example we choose muon pair production $e^+e^- \to \mu^+\mu^-$. Muon electron scattering is then left as an exercise. Let us begin by writing down the matrix element using the Feynman rules

$$\mathcal{A} =$$



$$= \bar{v}(p_1)(-\mathrm{i}e\gamma^\mu)u(p_2)\left(\frac{-\mathrm{i}g_{\mu\nu}}{q^2+\mathrm{i}0^+}\right)\bar{u}(p_3)(-\mathrm{i}e\gamma^\nu)v(p_4)\,.$$

We now need $|\mathcal{A}|^2 = \mathcal{A}\mathcal{A}^*$. Because $\mathcal{A}$ is a complex number $\mathcal{A}^* = \mathcal{A}^\dagger$ so let us just calculate that and see where it leads

$$\mathcal{A}^\dagger = \Big(\bar{v}_1(-\mathrm{i}e\gamma^{\mu'})u_2\Big)^\dagger\left(\frac{+\mathrm{i}g_{\mu'\nu'}}{q^2+\mathrm{i}0^+}\right)\Big(\bar{u}_3(-\mathrm{i}e\gamma^{\nu'})v_4\Big)^\dagger\,.$$

For the spinor line we note that $\bar{u} = u^\dagger\gamma^0$ and that, under adjungation the order of matrices and vectors reverses

$$\Big(\bar{v}_1\gamma^{\mu'}u_2\Big)^\dagger = u_2^\dagger(\gamma^{\mu'})^\dagger(\gamma^0)^\dagger v_1\,.$$

Using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, $(\gamma^0)^\dagger = \gamma^0$ as well as $\gamma_0^2 = 1$ we can re-write this as

$$\Big(\bar{v}_1\gamma^{\mu'}u_2\Big)^\dagger = u_2^\dagger\gamma^0\gamma^{\mu'}\gamma^0\gamma^0 v_1 = \bar{u}_2\gamma^{\mu'}v_1$$

and similarly for the other line. We now have

$$\begin{aligned}|\mathcal{A}|^2 &= (-\mathrm{i}e)^2(-\mathrm{i})(+\mathrm{i}e)^2(+\mathrm{i})\frac{1}{q^4}\Big(\bar{v}_1\gamma^\mu u_2\ \bar{u}_2\gamma^{\mu'}v_1\Big)\Big(\bar{u}_3\gamma_\mu v_4\ \bar{v}_4\gamma_{\mu'}u_3\Big)\,.\\ &= \frac{e^4}{q^4}\Big((\bar{v}_1)_\alpha\gamma^\mu_{\alpha\beta}(u_2)_\beta\ (\bar{u}_2)_\delta\gamma^{\mu'}_{\delta\rho}(v_1)_\rho\Big)\Big((\bar{u}_3)_\alpha\gamma^\mu_{\alpha\beta}(v_4)_\beta\ (\bar{v}_4)_\delta\gamma^{\mu'}_{\delta\rho}(u_3)_\rho\Big)\,.\end{aligned}$$

We now note that v and u form a completeness relation

$$\sum_s u^s_\alpha(p)\bar{u}^s_\beta(p) = (\not{p}+m)_{\alpha\beta} \qquad \text{and} \qquad \sum_s v^s_\alpha(p)\bar{v}^s_\beta(p) = (\not{p}-m)_{\alpha\beta}\,. \tag{12}$$

This means we could simplify $|\mathcal{A}|^2$ by summing over final state spins and averaging over initial states (assuming the experiment does not measure / prepare these quantities)

$$\begin{aligned}|\mathcal{A}|^2 &= \frac{e^4}{q^4}\Big((\bar{v}_1)_\alpha\gamma^\mu_{\alpha\beta}(\not{p}_2+m)_{\beta\delta}\gamma^{\mu'}_{\delta\rho}(v_1)_\rho\Big)\Big((\bar{u}_3)_\alpha\gamma^\mu_{\alpha\beta}(\not{p}_4-M)_{\beta\delta}\gamma^{\mu'}_{\delta\rho}(u_3)_\rho\Big)\,.\\ &= \frac{e^4}{q^4}\Big((\not{p}_1-m)_{\rho\alpha}\gamma^\mu_{\alpha\beta}(\not{p}_2+m)_{\beta\delta}\gamma^{\mu'}_{\delta\rho}\Big)\Big((\not{p}_3-M)_{\rho\alpha}\gamma^\mu_{\alpha\beta}(\not{p}_4-M)_{\beta\delta}\gamma^{\mu'}_{\delta\rho}\Big)\,.\end{aligned}$$

This is now the definition of a trace in spinor space $A_{\rho\rho} = \mathrm{tr}A$

$$|\mathcal{A}|^2 = \frac{e^4}{4q^4}\mathrm{tr}\Big((\not{p}_1-m)\gamma^\mu(\not{p}_2+m)\gamma^{\mu'}\Big)\mathrm{tr}\Big((\not{p}_3-M)\gamma_\mu(\not{p}_4+M)\gamma_{\mu'}\Big)\,.$$

These are now objects we can calculate easily. By using (11).

We need identities for traces of γ matrices.

- $\mathrm{tr}\gamma^\nu = 0$: Use (11) with $\mu = \nu$, i.e. $\gamma^\mu\gamma_\mu = 4$

$$\mathrm{tr}(\gamma^\nu) = \frac{1}{4}\mathrm{tr}(\gamma^\nu\gamma^\mu\gamma_\mu) \overset{*}{=} -\frac{1}{\mathrm{tr}}(\gamma^\mu\gamma^\nu\gamma_\mu) = -\frac{1}{4}\mathrm{tr}(\gamma^\nu\gamma^\mu\gamma_\mu)\,,$$

  where we used (11) again at $*$.

- traces of odd numbers of γ matrices vanish.

- $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$. Using (11) and cyclicity

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\Big(\mathrm{tr}(\gamma^\mu\gamma^\nu) + \mathrm{tr}(\gamma^\nu\gamma^\mu)\Big) = \frac{1}{2}\mathrm{tr}\{\gamma^\mu,\gamma^\nu\} = g^{\mu\nu}\mathrm{tr}1 = 4g^{\mu\nu}$$

- $\mathrm{tr}\Big(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\Big) = 4\Big(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\Big)$ similarly

For $0 \sim m \ll M$ we write

$$\begin{aligned}
|\mathcal{A}|^2 &= \frac{e^4}{4q^4}\mathrm{tr}\Big(\not{p}_1\gamma^\mu\not{p}_2\gamma^{\mu'}\Big)\Big[\mathrm{tr}\Big(\not{p}_3\gamma_\mu\not{p}_4\gamma_{\mu'}\Big) + M\cancel{\mathrm{tr}\Big(\not{p}_3\gamma_\mu\gamma_{\mu'}\Big)} - M\cancel{\mathrm{tr}\Big(\gamma_\mu\not{p}_4\gamma_{\mu'}\Big)} - M^2\mathrm{tr}\Big(\gamma_\mu\gamma_{\mu'}\Big)\Big] \\
&= \frac{4e^4}{q^4}\Big(p_1^\mu p_2^{\mu'} - p_1\cdot p_2 g^{\mu\mu'} + p_1^{\mu'}p_2^\mu\Big)\Big[\Big(p_3^\mu p_4^{\mu'} - p_3\cdot p_4 g^{\mu\mu'} + p_3^{\mu'}p_4^\mu\Big) - M^2 g^{\mu\mu'}\Big] \\
&= \frac{8e^4}{q^4}\Big(M^2 p_1\cdot p_2 + p_1\cdot p_4 p_2\cdot p_3 + p_1\cdot p_3 p_2\cdot p_4\Big)\,.
\end{aligned}$$

Using the Mandelstam variables s, t and u as well as $q = p_1 + p_2$ we can write

$$p_1\cdot p_2 = \tfrac{s}{2} p_1\cdot p_3 = p_2\cdot p_4 = \tfrac{1}{2}(M^2 - t) p_1\cdot p_4 = p_2\cdot p_3 = \tfrac{1}{2}(M^2 - u)$$

we can now write down the matrix element in terms of experimentally accessible variables

$$|\mathcal{A}|^2 = \frac{4e^4}{s^2}\left(\frac{1}{2}(t^2 - u^2) + M^2(s - t - u) + M^4\right) \to \frac{2e^4(t^2 + u^2)}{s^2}\,.$$

The differential cross section in the high-energy limit is now with $s + t + u = 0$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi s^2}|\mathcal{A}|^2 = 2\pi\alpha^2\frac{s^2 + 2st + 2t^2}{s^4}\sigma = \int_0^s \mathrm{d}t\,\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{16\pi\alpha^2}{3s}\,.$$

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